DYNAMIC MODELING

INTRODUCTION

A critical step in the analysis and design of a mechatronic system is the acquisition, by some means, of a mathematical model of the system of interest. Most often this requires the derivation of the model from first principles, or the modification of an existing model. Typically the model consists of a set of ordinary differential equations (ODE's), possibly nonlinear, which are said to "govern" the behavior of the system, hence the name Governing Equations, or Truth Model. It must sufficiently capture the system's behavioral characteristics as to enable an accurate performance analysis and the identification of any performance problems that may exist in the design.

The Design Model, on the other hand, is used for the purpose of controller design. It needs therefore to capture only those characteristics of the system necessary to enable an adequate controller design. It can be linear and time-invariant in order to allow application of the more tractable linear control design techniques; or it may be necessary in order to meet the desired performance limits, to include time dependencies and/or nonlinearities in the Design Model.

There are many widely varied areas of study that are pertinent to the development of mathematical models for control. The focus here is primarily on mechanical systems modeling – Newton's laws for linear and angular motion, friction and other nonlinearities, gears and belt drives. Also, briefly covered are the basics of modeling thermal systems. It is not an objective of this text to provide a broad, comprehensive coverage of the modeling of a wide range of physical systems. The reader can gather from other sources the information and data needed to create a model for their application and to that apply the model-based design techniques discussed herein.

The primary topics covered in this chapter therefore include:

- conversion of ODE's to transfer function and state-space forms
- mechanical system modeling
  - generation of governing equations for mechanical systems via Newton's 2nd Law
  - mechanical actuation via gears and belts
  - nonlinearities

It is assumed that the reader has an undergraduate background in linear systems, Laplace- and z- transforms, as well as familiarity with Matlab and Simulink.

TRANSFER FUNCTION AND STATE-SPACE MODELS

It is often the case that a dynamic model is needed in a particular form. For example, state-space form is preferred and best suited for use in doing a modern control type of control system type design. In this section we describe methods for conversion of ODE's to state-space or transfer function form. This is depicted in Figure 2-1 below. In the case of linear systems it is possible to convert from each of these forms to any of the others. The pathways most often required of the mechatronics engineer are those depicted by the arrows. A set of ODE's is derived from first principles. If nonlinear, it may be possible to retain those nonlinearities and bring them directly into the state-space model. Or if one is interested in the frequency response characteristics of the system, it can be linearized, converted to transfer function form, and from that a frequency response can be derived. Often in data sheets the characteristics of a sensor or actuator are provided in the form of a frequency response. It then becomes necessary to work back from a magnitude and phase curve set to identify the transfer function generating that response, thus the arrow from FR to TF. In generating a Design Model, that transfer function can be converted to its equivalent state-space form and then inserted into the state-space model of the overall system for which a controller is being designed.

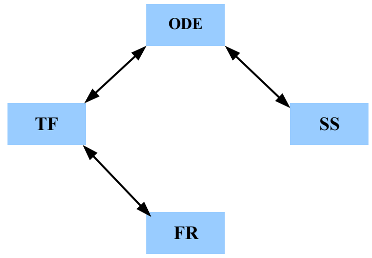
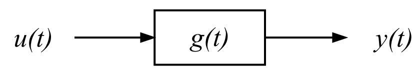


Figure 2-1 – Conversion of models to various forms: TF – Transfer Function, SS – State Space, ODE – Ordinary Differential Equation, FR – Frequency Response

Conversion from ODE to TF forms – Consider the n[th] order ODE representing the input-output relationship of linear system *G(s)*, where *G(s)* is the Laplace Transform of impulse response *g(t)*: $G(s) = L\big(g(t)\big)$.



$$G(s) = \frac{Y(s)}{U(s)} \tag{2.1}$$

The n^{th} order ordinary differential equation in general form can be written as:

$$\begin{aligned} \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y = \\ b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u \end{aligned} \tag{2.2}$$

with *n* coefficients multiplying derivatives of the output *y(t)*, and *m* coefficients multiplying derivatives of the input *u(t)*. The transfer function is obtained by taking the *y(t)* Laplace Transform of both sides with all pertinent initial conditions set to zero:

$$\left\{\frac{d^{n-1}y(0)}{dt^{n-1}}, \frac{d^{n-2}y(0)}{dt^{n-2}}, \ldots, \frac{dy(0)}{dt}, y(0), \frac{d^{m-1}u(0)}{dt^{m-1}}, \frac{d^{m-2}u0)}{dt^{m-2}}, \ldots, \frac{du(0)}{dt}, u(0)\right\} = 0_{(m+n)}$$

yielding:

$$\left\{s^n + a_{n-1}s^{n-1`} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0\right\}Y(s) = \left\{b_m s^{m`} + b_{m-1}s^{m-1`} + \cdots + b_1 s^{1`} + b_0\right\}U(s)$$

and transfer function:

$$\frac{Y(s)}{U(s)} = \frac{\left(b_m s^{m`} + b_{m-1}s^{m-1`} + \cdots + b_1 s^{1`} + b_0\right)}{\left(s^n + a_{n-1}s^{n-1`} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0\right)} \tag{2.3}$$

When in the form of Eq. **Error! Reference source not found.** its conversion to the form of an ordinary differential equation **Error! Reference source not found.** is straightforward.

Often the dot notation or the superscript notation is used to unclutter the ODE's. This notation is as follows:

$$\frac{dy}{dt} = \dot{y}, \quad \frac{d^2y}{dt^2} = \ddot{y}, \quad \frac{d^3y}{dt^3} = \dddot{y} = D^3y, \quad \frac{d^4y}{dt^4} = D^4y, \quad \frac{d^5y}{dt^5} = D^5y \cdots$$

It is employed in the example problems that follow.

EXAMPLES OF CONVERSION OF ODES TO TFS:

Ex, 1 $\quad \ddot{y}(t) + 21\dot{y}(t) + 20y(t) = u(t)$

Take Laplace Transform with $\dot{y}(0) = y(0) = 0$

$$(s^2 + 21s + 20)\,Y(s) = U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 21s + 20}$$

$$= \frac{1}{(s+1)(s+20)}$$

Ex, 2 $\quad \dddot{y} + 21\ddot{y} + 20\dot{y} = u$

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+1)(s+20)}$$

Ex 3

$$\frac{d^4y}{dt^4} + (2\xi\Omega_0 + \alpha)\frac{d^3y}{dt^3} + (2\xi\Omega_0\alpha + \Omega_0^2)\frac{d^2y}{dt^2} + \Omega_0^2\frac{dy}{dt} = Ku$$

$$\frac{Y(s)}{U(s)} = \frac{K}{s^4 + (2\xi\Omega_0 + \alpha)s^3 + (2\xi\Omega_0\alpha + \Omega_0^2)s^2 + \Omega_0^2 s}$$

$$= \frac{K}{s(s^2 + 2\xi\Omega_0 s + \Omega_0^2)(s+\alpha)}$$

Ex. 4 $\quad J \dfrac{d\omega}{dt} + B\omega = \tau$

$$W(s) = \mathcal{L}\{\omega(t)\} \qquad T(s) = \mathcal{L}\{\tau(t)\}$$

$$\frac{W(s)}{T(s)} = \frac{1}{Js + B}$$

Ex. 5 $\quad LJ\dddot{y} + (RJ + LB)\ddot{y} + (RB + K_T K_B)\dot{y} = K_T u$

$$\frac{Y(s)}{U(s)} = \frac{K_T}{s\left(LJs^2 + (RJ + LB)s + RB + K_T K_B\right)}$$

$$= \frac{K_T}{s\left[(Ls + R)(Js + B) + K_T K_B\right]}$$

Conversion from ODE and TF form to SS form – When in state variable form the system is represented not as a single n^{th} order differential equation in *y(t)* and *u(t)*, but as a set of *n* first-order, coupled equations, each governing the evolution of one state variable for a system again having input *u(t)* and output *y(t)*. The dimensions of the state vector *x(t),* input vector *u(t)*, and output vector *y(t)* are:

$$\begin{aligned} x(t) &\quad - \quad n \times 1 \\ y(t) &\quad - \quad p \times 1 \\ u(t) &\quad - \quad m \times 1 \end{aligned}$$

Considering initially the transfer function involving no zero, and expressing it in ODE form:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u \tag{2.4}$$

Reorganizing to express the highest derivative as a function of *u* and of all lower derivatives in *y*:

$$\frac{d^n y}{dt^n} = b_0 u - a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} - a_{n-2}\frac{d^{n-2}y}{dt^{n-2}} - \cdots - a_1\frac{dy}{dt} - a_0 y \tag{2.5}$$

A string of (n-1) integrators is constructed in Figure 2-2 and a block diagram representation of the general ODE involving only poles is produced:

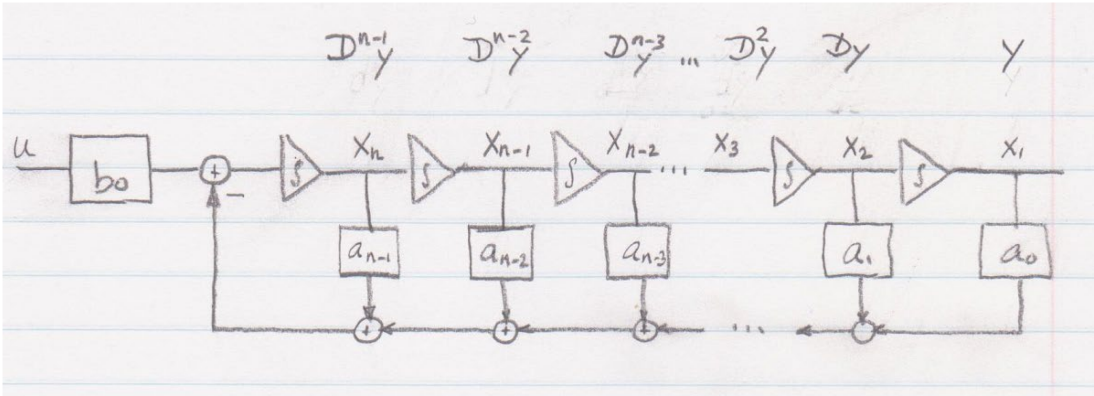


Figure 2-2 – Block diagram representation of the general n^{th} order ODE involving no derivatives of u

Assigning a state variable to the output of each integrator produces one realization of the state-space form of **Error! Reference source not found.**:

$$x_1 = y \qquad x_2 = \frac{dy}{dt} \qquad x_3 = \frac{d^2 y}{dt^2} \qquad \cdots \qquad x_{n-1} = \frac{d^{n-2} y}{dt^{n-2}} \qquad x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

Expressing these in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \overline{x}$$

State variables are not unique. Different state variable choices can have the same input-output transfer function characteristics and therefore represent the same system. It may often occur that the first three derivatives may be the physical variables of position, velocity, and acceleration. Higher derivatives expressed in this canonical form described above are not physical variables but derivatives of the acceleration. It may be preferable to use physical variables when possible, particularly when some of these are outputs that appear in the output vector y.

What can be done in the case in which zeros exist, and the coefficients $b_m \neq 0$ for $m \leq (n-1)$. In that case we decompose the numerator and denominator of **Error! Reference source not found.** as follows:

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{X(z)} \frac{X(s)}{U(z)}$$

$$\frac{Y(s)}{X(s)} = \left(b_m s^{m} + b_{m-1} s^{m-1} + \cdots + b_1 s^{1} + b_0\right)$$

$$\frac{X(s)}{U(s)} = \frac{1}{\left(s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0\right)}$$

We have already developed the block diagram realization of the transfer function from U to X, consisting of a string of integrators here involving successive derivatives of $x(t)$. The state vector associated with this portion of the system would be labeled x_1 through x_n.

Focusing attention on the transfer function from X to the output Y, we note that it requires summing a weighted sum of successive derivatives of $x(t)$. The output equation is therefore:

$$y(t) = b_m \frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1}x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x$$

which is readily constructed as shown in Figure 2-3 below.

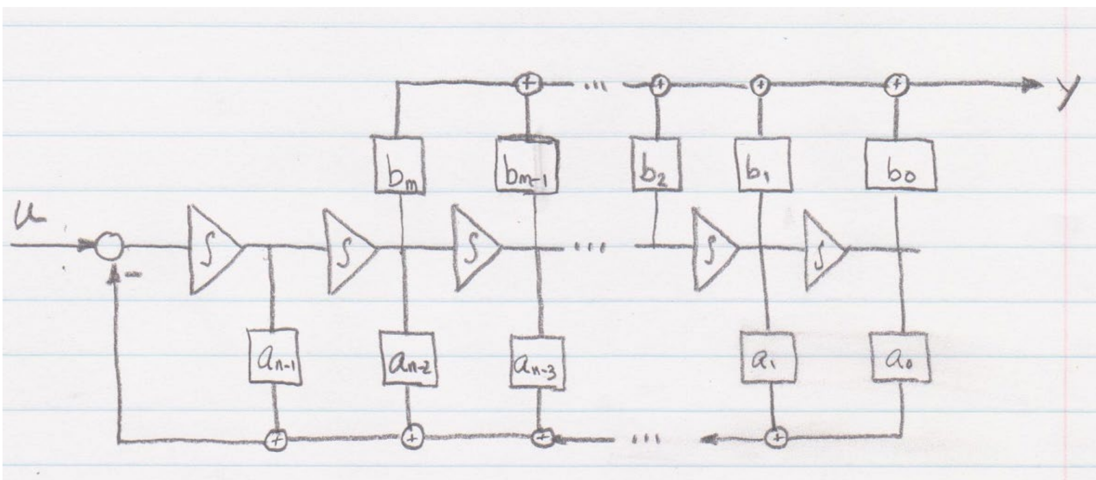


Figure 2-3 – Block diagram representation of the general n[th] order ODE involving both zero and poles, for the case in which the number of poles exceeds the number of zero by at least 1.

The corresponding state space realization is given as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & & b_2 & b_3 & \cdots & b_m & \end{bmatrix} \overline{x}$$

Note that this form of state-space formulation is not unique but depends on a number of things, one being the definition of state variables as noted. Another is the numbering of the states.

In the case of a single-input multi-output system, multiple rows in the output vector y are represented by additional summation chains containing the corresponding elements of each row b_{kj}. In the case of multi-input single output system it is also possible to generate the appropriate block diagram and state space formulation**.** The reader interested in this development is referred to (Friedland: Control System Design).

MECHANICAL SYSTEM MODELING

TRANSLATIONAL MOTION

This section provides a brief introduction to dynamic modeling of mechanical systems using Newton's 2^{nd} Law for translational and rotational motion. Newton's 2^{nd} Law for translational motion is as follows:

$$m\frac{dv}{dt} = \Sigma f$$

where the coefficient *m* is the mass of the body, variable *v(t)* the velocity of the body, and Σf the sum of forces being applied to the body.

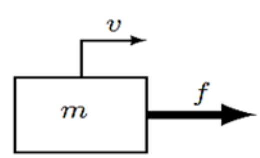
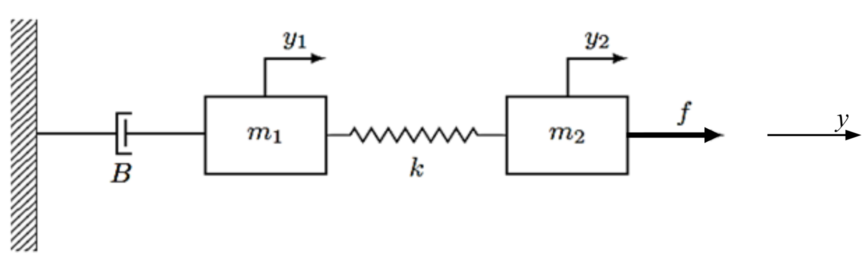


Figure 2-4 – Single degree of freedom mass in translation

The sum of all external forces acting on the body is equal to the mass times the acceleration. In this diagram we are depicting a body having a single degree of freedom, i.e. motion along a straight line. Other basic mechanical elements are the linear spring and damper. The force applied by a linear spring is governed by Hook's law which state that the force applied by the spring is equal to the spring constant *k* times the displacement (i.e. stretching or compression) of one end of the spring relative to the other. The constant *k* has units of force per unit distance; e.g. Newton's per meter. The damper also applies a force when the ends of the damper move relative to one another. The damping force is proportional to the relative velocity of the damper's ends, and the proportionality constant is the damping coefficient *b*. This is also referred to as viscous damping.

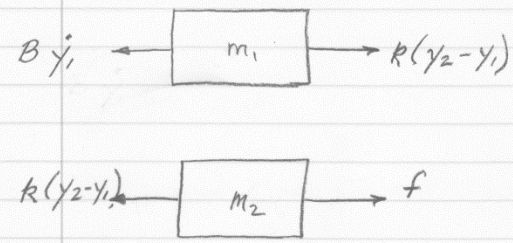
EXAMPLE 2-1 – TRANSLATIONAL MOTION, 2 BODY EXAMPLE

The 4^{th} order dynamic system shown here includes two bodies, each having only a single degree of freedom, motion along the axis depicted by the vector denoted with the label *y*.



For this system (1) derived the dynamic equations of motion, (2) a state space model, (3) transfer functions from f to y_1 and f to y_2, and (4) the frequency response of both transfer functions.

Begin by sketching the free body diagrams and deriving using Newton's 2nd law for translational motion the governing equations of motion:



$$m_1\ddot{y}_1 = k(y_2-y_1) - B\dot{y}_1$$

$$m_2\ddot{y}_2 = f - k(y_2-y_1)$$

$$m_1\ddot{y}_1 + B\dot{y}_1 + ky_1 = ky_2$$

$$m_2\ddot{y}_2 + ky_2 = f + ky_1$$

Develop the state-space formulation:

Let $x_1 = y_1$ $\quad x_3 = y_2$

$x_2 = \dot{y}_1$ $\quad x_4 = \dot{y}_2$

Then $\dot{x}_1 = x_2$ $\quad \dot{x}_3 = x_4$

From governing equations:

$$m_1\dot{x}_2 + Bx_2 + kx_1 = kx_3$$

$$m_2\dot{x}_4 + kx_3 = f + kx_1$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -k/m_1 & -B/m_1 & k/m_1 & 0\\ 0 & 0 & 0 & 1\\ k/m_2 & 0 & -k/m_2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}f$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

Derive the transfer functions:

Laplace transform both:

$$(m_1 s^2 + Bs + k) y_1 = k y_2 \quad (1)$$

$$(m_2 s^2 + k) y_2 = f + k y_1 \quad (2)$$

Substitute (1) into (2)

$$(m_2 s^2 + k)(m_1 s^2 + Bs + k)\frac{1}{k} y_1 = f + k y_1$$

$$\left[(m_2 s^2 + k)(m_1 s^2 + Bs + k) - k^2\right] y_1 = k f$$

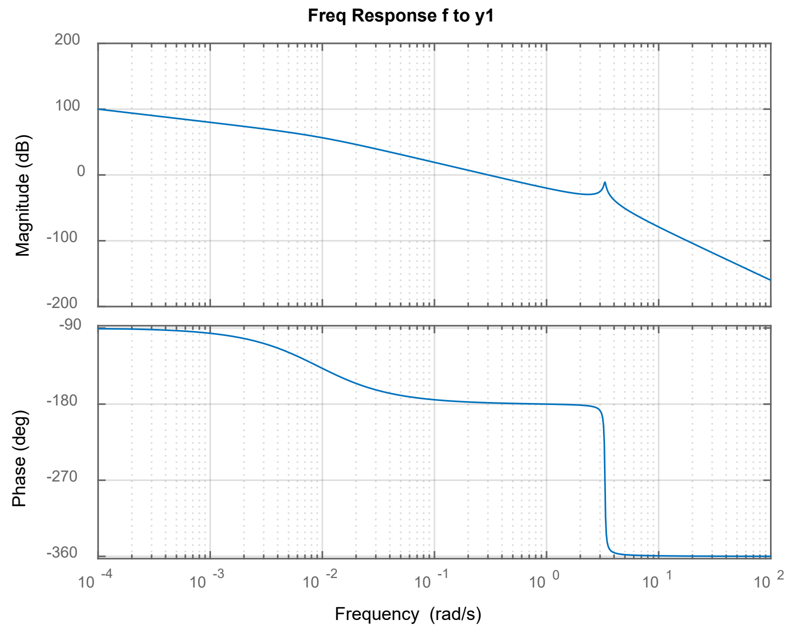
$$\boxed{\frac{y_1}{f} = \frac{k}{(m_2 s^2 + k)(m_1 s^2 + Bs + k) - k^2}}$$

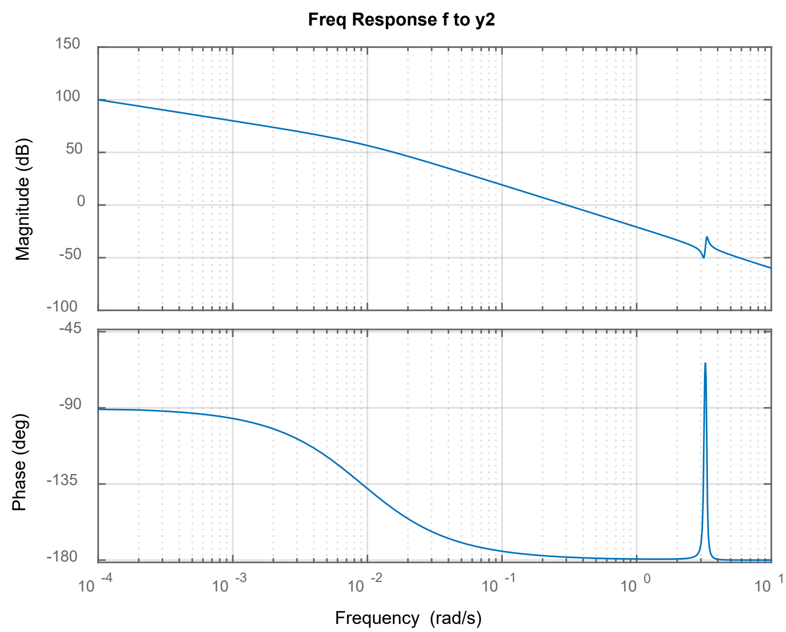
Since $\frac{y_2(s)}{y_1(s)} = \frac{m_1 s^2 + Bs + k}{k}$

$$\boxed{\frac{y_2}{f} = \frac{m_1 s^2 + Bs + k}{(m_2 s^2 + k)(m_1 s^2 + Bs + k) - k^2}}$$

Use Matlab to compute the transfer functions numerically, and plot the bode plots. Note that the convolve function [conv(A, B)] performs the polynomial multiplication of the polynomials represented by polynomial vectors A and B.

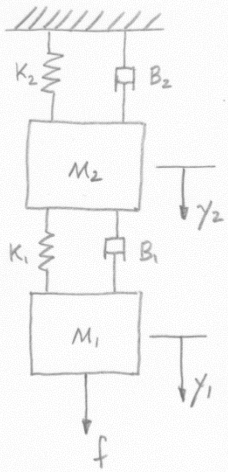
```
% Example 2-1
m2 = 10;
m1 = 1;
k = 10;
B = 0.1;

% demonimator and numerator of f to y2
den = conv([m2 0 k], [m1 B k]) - [ 0 0 0 0 k^2];
num = [m1 B k];

% Frequency Response f to y1
figure(1), bode(k, den), title('Freq Response f to y1'), grid

% Frequency Response f to y2
figure(2), bode(num, den), title('Freq Response f to y2'), grid
```


Freq Response f to y1
Magnitude (dB)
200
100
0
-100
-200
Phase (deg)
-90
-180
-270
-360
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)


Freq Response f to y2
Magnitude (dB)
150
100
50
0
-50
-100
Phase (deg)
-45
-90
-135
-180
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequency (rad/s)

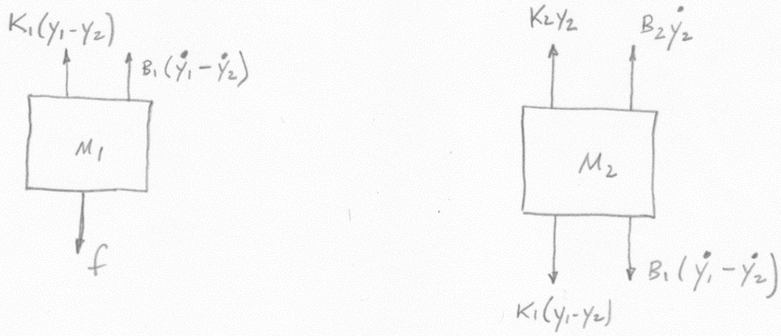
EXAMPLE 2-2 – TRANSLATIONAL MOTION, 2 BODY EXAMPLE WITH GRAVITY

We begin by examining the case in which there is no gravity.



- Assumptions – No gravity forces
- $y_1 = 0$ & $y_2 = 0$ when $f = 0$

Free-body diagrams



Apply Newton's 2nd Law: $\Sigma F = ma$

$$\Sigma F_1 = f - K_1(y_1 - y_2) - B_1(\dot{y}_1 - \dot{y}_2)$$

$$\boxed{M_1\ddot{y}_1 + B_1(\dot{y}_1 - \dot{y}_2) + K_1(y_1 - y_2) = f} \tag{2.6a}$$

$$\Sigma F_2 = K_1(y_1 - y_2) + B_1(\dot{y}_1 - \dot{y}_2) - K_2 y_2 - B_2\dot{y}_2$$

$$\boxed{M_2\ddot{y}_2 - B_1(\dot{y}_1 - \dot{y}_2) + B_2\dot{y}_2 - K_1(y_1 - y_2) + K_2 y_2 = 0} \tag{2.6b}$$

(2)

Transfer Functions $\frac{Y_1}{f}$ $\frac{Y_2}{f}$ where $Y_1 = Y_1(s)$ $f = f(s)$
$Y_2 = Y_2(s)$

Taking Laplace of ODE's:

(1) $$(M_1 s^2 + B_1 s + K_1) Y_1 - (B_1 s + K_1) Y_2 = f$$

(2) $$[M_2 s^2 + (B_2 + B_1) s + (K_2 + K_1)] Y_2 - (B_1 s + K_1) Y_1 = 0$$

from (2) $$\frac{Y_2}{Y_1} = - \frac{B_1 s + K_1}{M_2 s^2 + (B_2 + B_1) s + K_2 + K_1} \qquad (3)$$

substituting this into (1)

$$(M_1 s^2 + B_1 s + K_1) Y_1 - \frac{(B_1 s + K_1)^2}{M_2 s^2 + (B_2 + B_1) s + (K_2 + K_1)} Y_1 = f$$

$$\frac{\left[(M_1 s^2 + B_1 s + K_1)(M_2 s^2 + (B_2 + B_1) s + (K_2 + K_1)) - (B_1 s + K_1)^2\right] Y_1}{M_2 s^2 + (B_2 + B_1) s + (K_2 + K_1)} = f$$

$$\boxed{\frac{Y_1}{f} = \frac{M_2 s^2 + (B_2 + B_1) s + (K_2 + K_1)}{(M_1 s^2 + B_1 s + K_1)(M_2 s^2 + (B_2 + B_1) s + (K_2 + K_1)) - (B_1 s + K_1)^2}}$$

This could be further simplified to group all of the powers of s, however this is sufficient.

Using eq. (3):

$$\boxed{\frac{Y_2}{f_1} = \frac{B_1 s + K_1}{(M_1 s^2 + B_1 s + K_1)(M_2 s^2 + (B_2 + B_1) s + (K_2 + K_1)) - (B_1 s + K_1)^2}}$$

3

State-Space Formulation

Let
$$x_1 = y_1$$
$$x_2 = \dot{y}_1 = \dot{x}_1$$
$$x_3 = y_2$$
$$x_4 = \dot{y}_2 = \dot{x}_3$$

Substitute into the diff. eq. of motion

$$M_1 \dot{x}_2 + B_1 (x_2 - x_4) + K_1 (x_1 - x_3) = f$$
$$M_2 \dot{x}_4 - B_1 (x_2 - x_4) + B_2 x_4 - K_1 (x_1 - x_3) + K_2 x_3 = 0$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_3 = x_4$$

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} \quad x_1 \quad & \quad x_2 \quad & \quad x_3 \quad & \quad x_4 \quad \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K_1/M_1 & -B_1/M_1 & K_1/M_1 & B_1/M_1 \\ 0 & 0 & 0 & 1 \\ K_1/M_2 & B_1/M_2 & \dfrac{-(K_1+K_2)}{M_2} & \dfrac{-(B_1+B_2)}{M_2} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M_1 \\ 0 \\ 0 \end{bmatrix} f}$$

Now add gravity and a gravitational force to each block – M_1*g and M_2*g. The ODE's become:

$$\begin{aligned} M_1\ddot{y}_1 + B_1\dot{y}_1 + K_1 y_1 - B_1\dot{y}_2 - K_1 y_2 &= f + M_1 g \\ M_2\ddot{y}_2 + (B_1 + B_2)\dot{y}_2 + (K_1 + K_2)y_2 - B_1\dot{y}_1 - K_1 y_1 &= M_2 g \end{aligned} \tag{2.7}$$

The gravity terms – M_1*g and M_2*g – produce a static deflection and a new equilibrium point. To determine the equilibrium set all derivative terms and the force *f(t)* to zero in the ODE's, as this is the condition when at rest at the equilibrium:

$$K_1(y_1 - y_2) = M_1 g$$
$$(K_1 + K_2)y_2 - K_1 y_1 = M_2 g$$

Solve for the equilibrium point:

$$y_{2e} = \frac{(M_1 + M_2)g}{K_2}$$
$$y_{1e} = \frac{M_1}{K_1}g + \frac{(M_1 + M_2)}{K_2}g$$

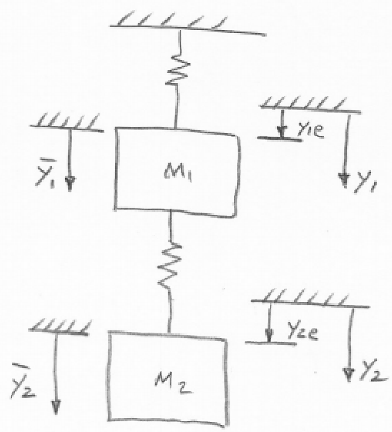
At the equilibrium the gravity forces equal the spring displacement forces in magnitude and are opposite in direction. Writing the ODEs in terms of new displacement variables that are relative to the equilibrium:

$$\bar{y}_1 = y_1 - y_{1e}$$
$$\bar{y}_2 = y_2 - y_{2e}$$

eliminates the gravity bias forces, resulting in the ODEs as given originally, without the gravity terms.

One computes the new ODE's and eliminates gravity terms as follows. We've defined new displacement variables relative to the equilibrium points:

$$\bar{y}_1 = y_1 - y_{1e}$$
$$\bar{y}_2 = y_2 - y_{2e}$$



Plug these into Eq. 2.7, noting that $\dot{\bar{y}}_1 = \dot{y}_1$, $\ddot{\bar{y}}_1 = \ddot{y}_1$, and so on.

$$M_1 \ddot{\bar{y}}_1 + B\dot{\bar{y}}_1 + K_1(\bar{y}_1 + y_{1e}) - B_1\dot{\bar{y}}_2 - K_1(\bar{y}_2 + y_{2e}) = f + M_1 g \qquad (*)$$

$$M_2\ddot{\bar{y}}_2 + (B_1+B_2)\dot{\bar{y}}_2 + (K_1+K_2)(\bar{y}_2+y_{2e}) - B_1\dot{\bar{y}}_1 - K_1(\bar{y}_1+y_{1e}) = M_2 g \qquad (\#)$$

Take these terms out: $y_{1e} - y_{2e} = \dfrac{M_1 g}{K_1} + \dfrac{(M_1+M_2)g}{K_2} - \dfrac{(M_1+M_2)g}{K_2}$

$$y_{1e}-y_{2e} = \frac{M_1 g}{K_1}$$

$$K_2\, y_{2e} = K_2 \frac{(M_1+M_2)}{K_2} g = (M_1+M_2)g$$

Ignoring the $\ddot{\bar{y}}_1$ and $\dot{\bar{y}}_1$ terms for the moment, Eq. * contains $K_1(y_{1e}-y_{2e}) = M_1 g$

which cancels the $M_1 g$ on the right side of the equal sign:

$$M_1\ddot{\bar{y}}_1 + B\dot{\bar{y}}_1 + K_1(\bar{y}_1-\bar{y}_2) - B\dot{\bar{y}}_2 + \cancel{K_1\frac{M_1 g}{K_1}} = f + \cancel{M_1 g} \quad (a)$$

Similarly in Eq #:

$$M_2\ddot{\bar{y}}_2 + (B_1+B_2)\dot{\bar{y}}_2 + (K_1+K_2)\bar{y}_2 - B_1\dot{\bar{y}}_1 - K_1\bar{y}_1 + K_2 y_{2e} + K_1(y_{2e}-y_{1e}) = M_2 g$$

But $K_2 y_{2e} + K_1(y_{2e}-y_{1e}) = (M_1+M_2)g - M_1 g = M_2 g$

Leaving

$$M_2\ddot{\bar{y}}_2 + (B_1+B_2)\dot{\bar{y}}_2 + (K_1+K_2)\bar{y}_2 - B\dot{\bar{y}}_1 - K_1\bar{y}_1 = 0 \quad (b)$$

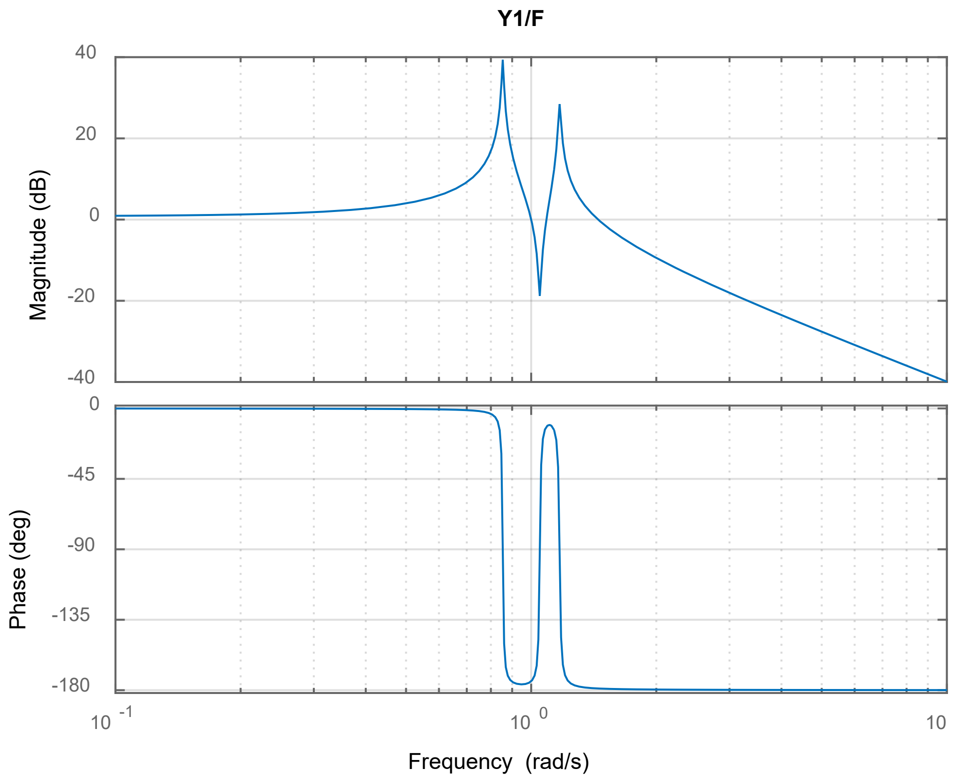
Eq's (a) and (b) are identical to Eq. 2.7 developed without gravity terms.

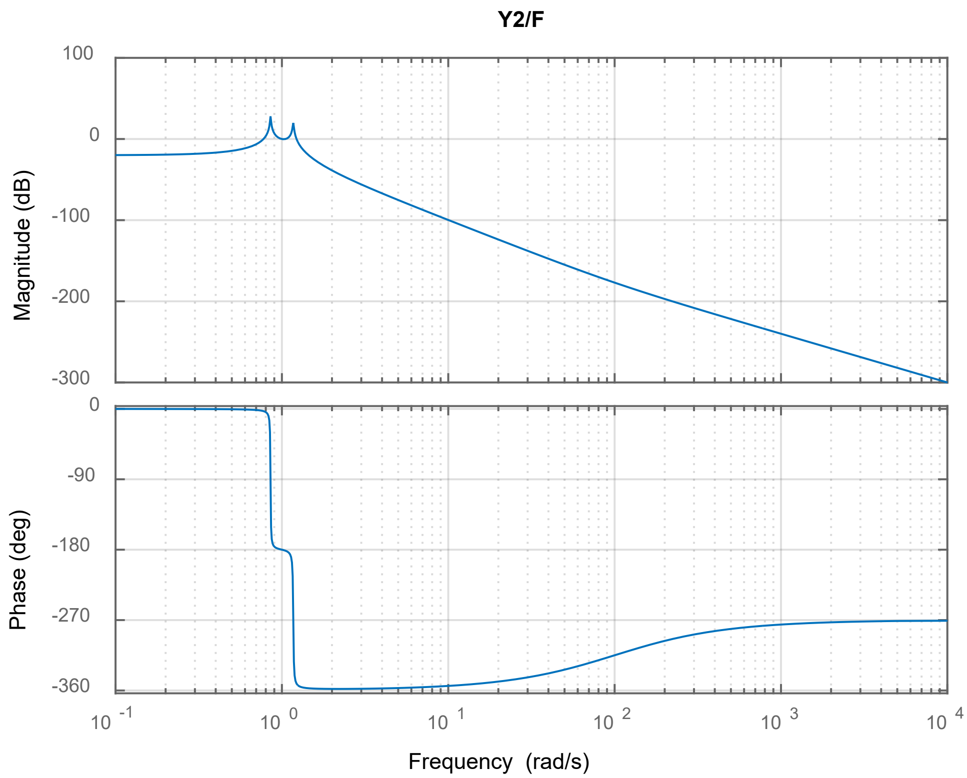
So for computing the transfer functions and frequency response we use the equations without gravity, and for simulation use the ODEs either with gravity or without, taking care to use the correct variables, the original or the displaced location variables, when defining the

Next we'll compute the frequency response for a particular set of parameters:

```
% Example 2-2
% Coefficients
M2=10;
M1=1;
K2=10;
K1=1;
B2=.1;
B1=0.01;

% Compute num and den polynomials
num = [B1 K1];
d1 = conv([M1 B1 K1], [M2 (B1+B2) (K1+K2)])
d2 = conv(num,num)
den = d1 - [0 0 d2]

% Plot bode plot -- Y2/F
```

```
bode(num,den), title('Y2/F')
grid

% similarly for Y1/F
num = [M2 (B2+B1) (K2+K1)];
figure, bode(num,den), title('Y1/F')
grid
```





ROTATIONAL MOTION

Newton's Law for rotational motion is $\sum T = J\ddot{\theta}$ – the sum of the torques acting on a body about an axis A-A equals the change in angular momentum of the body about that axis. For the case of constant mass moment of inertia J, the time rate of change of angular momentum is that inertia multiplied by its angular acceleration. We are considering a body having a single degree of freedom, having an axis of rotation A-A, having inertia J about that axis. A torque τ is applied to the body and the body has rotated through an angle θ relative to some datum location.

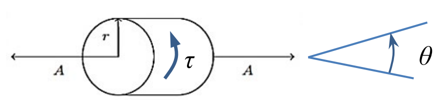
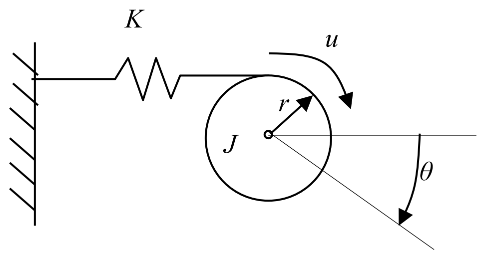


Figure 2-5 – Single degree of freedom mass in rotation

Computation of torques from forces applied to the body is done by multiplying the force and the distance between the force and the axis A-A, using the orthogonal distance between A-A and the force vector.

EXAMPLE 2-3: ROTATIONAL SPRING-MASS SYSTEM

Consider the dynamic system:



which is free to rotate about its center through angle θ about a frictionless pivot. The system has control input torque u, and disk angular position θ, disk inertia J, disk radius – r, damping coefficient – $\beta = 0$, and spring rate K. Assume that the spring force is zero when $\theta = 0$.

(a) Draw the free-body diagram showing all forces and torques acting on the disk (do not show forces acting at center pivot as they don't apply a torque to the disk).
(b) Derive the differential equation of motion.
(c) Cast this equation into state-space form, i.e.:

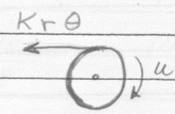
$$\dot{x} = Ax + Bu$$
$$y = Cx = z$$

clearly identifying A, B, and C in terms of constants K, r, and J. Use

$$x_1 = \dot{\theta}$$
$$x_2 = \theta$$

(d) Take the Laplace transform of the differential equations and derive the transfer function from u to θ; i.e. $\dfrac{\Theta(s)}{U(s)}$

(2) (a) FREE BODY DIAGRAM



(b) ODE by Newton's 2nd Law

$$\Sigma T = J\ddot{\theta}$$

$$u - (Kr\theta)r = J\ddot{\theta}$$

$$\boxed{J\ddot{\theta} + Kr^2\theta = u}$$

(c) $x_1 = \dot{\theta}$

$x_2 = \theta$

Thus $\dot{x}_2 = x_1$

$$\dot{x}_1 = \ddot{\theta} = (u - Kr^2\theta)/J$$

$$\dot{x}_1 = (1/J)u - (Kr^2/J)x_2$$

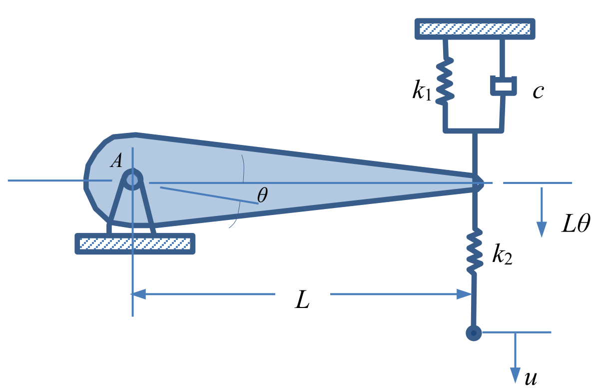
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 0 & -Kr^2/J \\ 1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/J \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1](x_1 \ x_2)^T$$

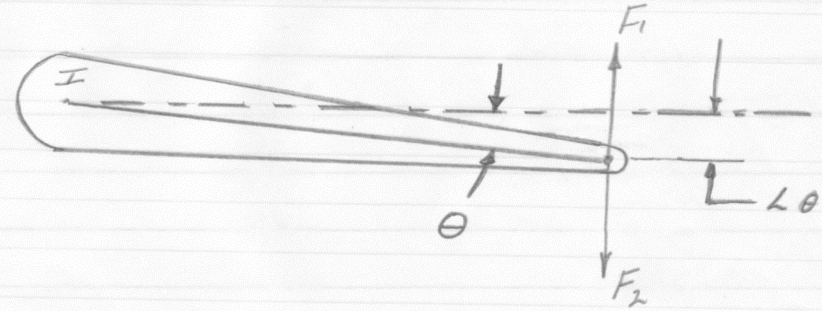
Laplace transform: $$\frac{U(s)}{\theta(s)} = \frac{1}{(Js^2 + Kr^2)}$$

EXAMPLE 2-4: ROTATIONAL SPRING-MASS SYSTEM, SMALL ANGULAR DISPLACEMENT

In this example, a rigid body is hinged at point A and moves in a horizontal plane (no gravity). For small displacement, obtain governing differential equation, transfer function, state space model and frequency response from control input displacement u to angle θ.



Drawing free body diagram and applying Newton's 2nd Law:



FROM Newton's Law $\quad \Sigma T = I\ddot{\theta} \quad$ (1)

FIRST define forces:

$$F_1 = k_1 L\theta + cL\dot{\theta}$$

$$F_2 = k_2(u - L\theta)$$

$$\Sigma T = F_2 L - F_1 L$$
$$= k_2 L(u - L\theta) - k_1 L^2\theta - cL^2\dot{\theta}$$

Applying (1): $\quad I\ddot{\theta} = k_2 L u - k_2 L^2\theta - k_1 L^2\theta - cL^2\dot{\theta}$

$$\boxed{I\ddot{\theta} + cL^2\dot{\theta} + (k_2 L^2 + k_1 L^2)\theta = k_2 L u}$$

Taking Laplace Transform:

$$\left(I s^2 + cL^2 s + K_2 L^2 + K_1 L^2\right)\theta(s) = K_2 L\, u(s)$$

$$\boxed{\frac{\theta(s)}{u(s)} = \frac{K_2 L}{I s^2 + cL^2 s + (K_2 + K_1)L^2}}$$

For bode plot

$I =$	10	Kg-m^2
$L =$	0.1	m
$c =$	1e4	N-s/m
$K_1 =$	2e6	N/m
$K_2 =$	2e6	N/m

State-space Form

$$x_1 = \theta$$
$$x_2 = \dot{x}_1 = \dot{\theta}$$
$$\dot{x}_2 = \ddot{\theta}$$

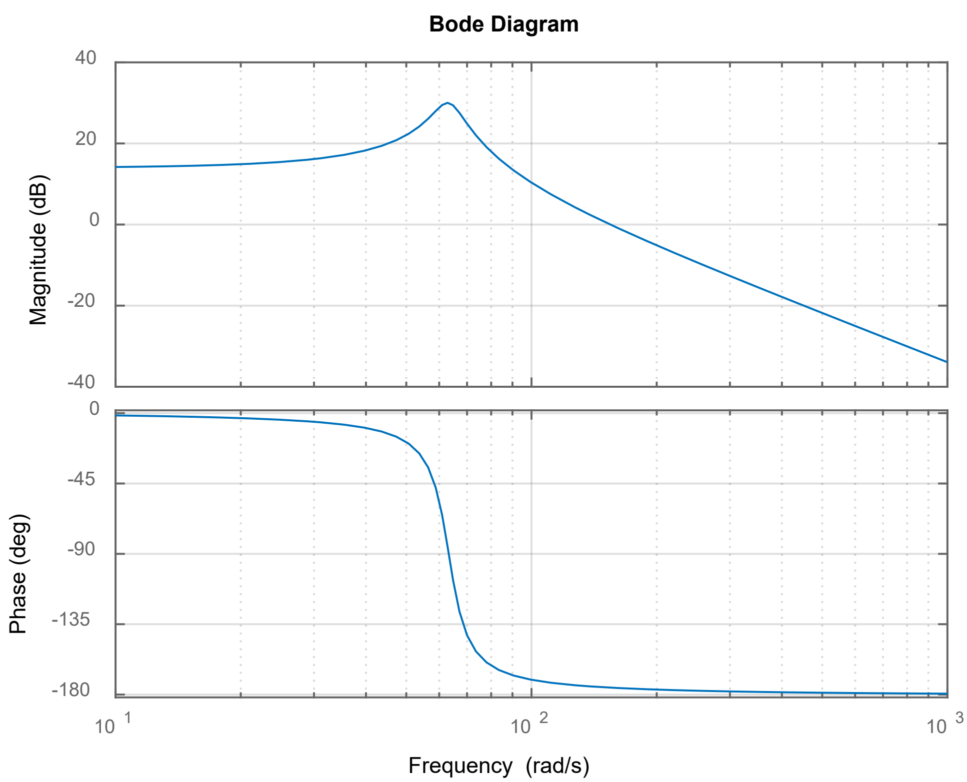
$$\dot{x}_2 = \frac{1}{I}\left[K_2 L u - (K_2 + K_1)L^2 x_1 - cL^2 x_2\right]$$
$$\dot{x}_1 = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(K_2+K_1)\frac{L^2}{I} & -c\frac{L^2}{I} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K_2 L/I \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

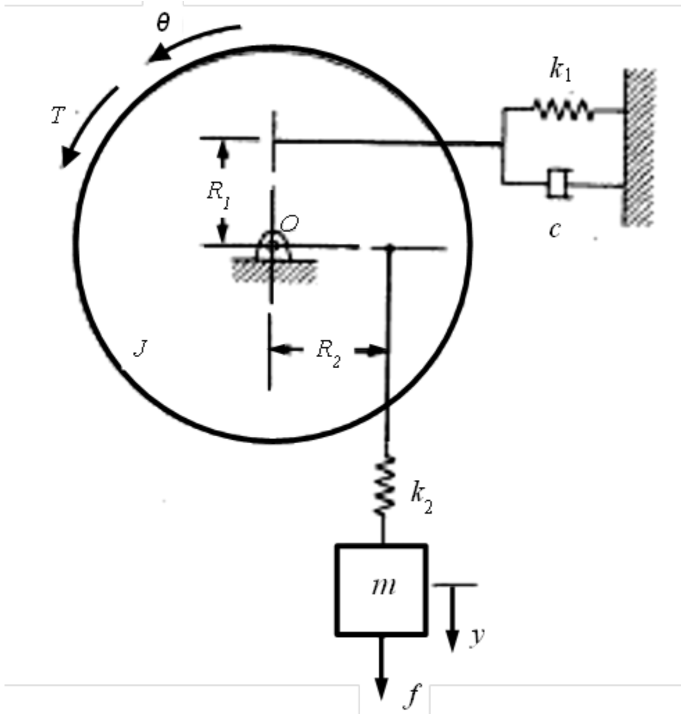
```
% Example 2-3
% Coefficients
I = 10;
L = 0.1;
c = 1e4;
k1 = 2e6;
k2 = 2e6;

% Compute num and den polynomials
num = k2*L;
den = [ I  c*L^2  (k2+k1)]

% Plot bode plot
bode(num,den)
grid
```

```
>> roots(den)

  -5.0000 +63.0476i
  -5.0000 -63.0476i
```

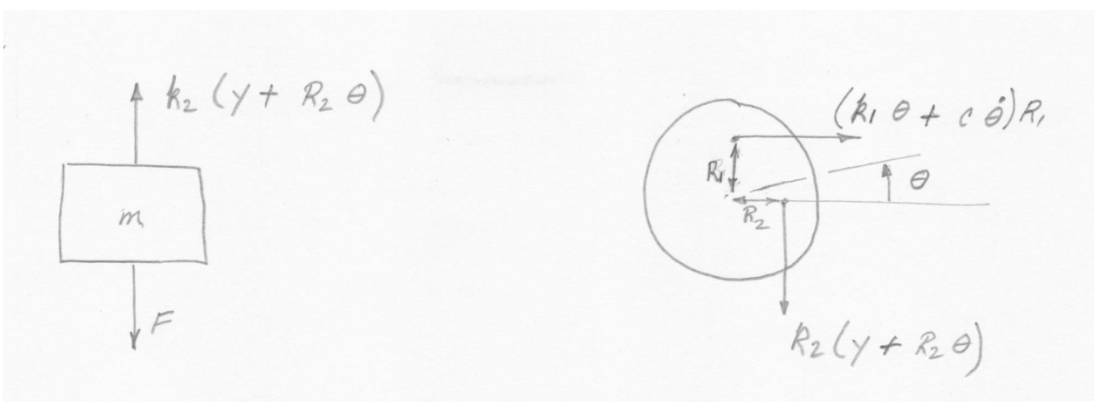


Example 2-5: Rotational and Translational Motion

A rigid body (disk) with moment of inertia J rotates around pivot point O. A block with mass m is attached to the disk through a spring. There are no gravity forces applied (i.e. the disk and block move horizontally and gravity is perpendicular to the plane of the paper.) Torque T and force f are applied to the disk and mass, respectively. For small displacements, obtain the governing differential equation and state-space model.



Free Body Diagrams:



$$\Sigma F = ma \quad \rightarrow \quad F - k_2(y+R_2\theta) = m\ddot{y}$$

$$\Sigma T = J\alpha \quad \rightarrow \quad -k_2(y+R_2\theta)R_2 - (k_1\theta + c\dot{\theta})R_1^2 = J\ddot{\theta}$$

Dynamic Equations:

$$m\ddot{y} + k_2 y + k_2 R_2 \theta = F$$

$$J\ddot{\theta} + k_1 c R_1^2 \dot{\theta} + (k_1 R_1^2 + k_2 R_2^2)\theta + k_2 R_2 y = 0$$

Transfer Function:

$$(m s^2 + k_2) y + k_2 R_2 \theta = F$$

$$\left[J s^2 + k_1 c R_1^2 s + (k_1 R_1^2 + k_2 R_2^2)\right]\theta + k_2 R_2 y = 0$$

Substitute 2nd equation into 1st to eliminate *y:*

$$-(m s^2 + k_2)\frac{(J s^2 + k_1 c R_1^2 s + (k_1 R_1^2 + k_2 R_2^2))\theta}{k_2 R_2} + k_2 R_2 \theta = F$$

$$\frac{(k_2 R_2)^2 - (m s^2 + k_2)(J s^2 + k_1 c R_1^2 s + (k_1 R_1^2 + k_2 R_2^2))}{k_2 R_2}\theta = F$$

$$\boxed{\frac{\theta}{F} = \frac{-k_2 R_2}{(m s^2 + k_2)(J s^2 + k_1 c R_1^2 s + k_1 R_1^2 + k_2 R_2^2) - (k_2 R_2)^2}}$$

State-space Formulation:

$$m\dot{x}_4 = -k_2 x_2 - k_2 R_2 x_1 + F$$

$$J\dot{x}_3 = -k_1 c R_1^2 x_3 - (k_1 R_1^2 + k_2 R_2^2)\, x_1 - k_2 R_2\, x_2$$

$$\boxed{\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K/J & -k_2 R_2/J & -k_1 c R_1^2/J & 0 \\ -k_2 R_2/m & -k_2/m & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m \end{bmatrix} F \qquad \text{where } K = k_1 R_1^2 + k_2 R_2^2}$$

TO BE ADDED:

System modeling

- Nonlinearities
- Mechanical actuation via gears and belts