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# PERFORMANCE SPECIFICATION

## INTRODUCTION

Control system designs often begin with the development and documentation of the relevant performance specifications. These can include (1) performance required under static conditions and during the application of static exogenous inputs (i.e. loads), (2) dynamic performance requirements defining (a) how accurately commanded inputs are to be tracked and (b) the system's response to time-varying disturbance inputs, and (3) the space of conditions (i.e. parameter variation) over which these requirements must be met. We examine this through the development of static and dynamic error budgets, the definition of time-domain transient response specifications, and frequency domain response specifications.

The outline of this chapter is:

- **Performance requirements & error budgets**
- **Time-domain specifications**
- **Frequency domain specifications and analysis**
  - bode plot methods
  - set-point tracking
  - disturbance rejection

## PERFORMANCE REQUIREMENTS – ERROR BUDGETS

Error budgets are the focus of an example in class.

## TIME-DOMAIN SPECIFICATIONS

Often in control system design, the designer is adjusting the control system to achieve particular pole locations. Thus, the designer must know how to define where those desired pole locations should be. Important time domain specifications are the rise time  $T_r$ , settling time  $T_s$ , and percent overshoot  $PO$ . This section examines how pole location affects these system performance characteristics. We examine this specifically for second-order systems. Why? Because the step response of stable systems that are higher than second-order commonly have a step response that is strongly influenced by the slowest eigenvalues in the system, e.g. the dominant conjugate symmetric poles closest to the origin. A performance specification limiting one or more of these three time-domain characteristics is used to define the location of the dominant, slowest complex pole pair, allowing the designer to focus on placing those poles with the control system being designed.

We begin by reviewing the relationship between pole location and dynamic response. Recall that the transfer function of a linear system is given by the Laplace Transform of its impulse response  $h(t)$ :

$$H(s) = L(h(t))$$

$H(s)$  can be represented as a ratio of polynomials in  $s$ :  $H(s) = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s}$ , having poles and zeros,  $p_i$  and  $z_i$ . In factored form  $H(s)$  can be expressed as follows:

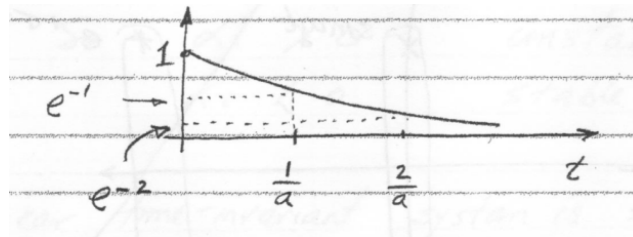
$$H(s) = K \frac{\prod (s - z_i)}{\prod (s - p_i)} = K \frac{(s - z_1)(s - z_2)(s - z_3) \cdots (s - z_m)}{(s - p_1)(s - p_2)(s - p_3) \cdots (s - p_n)}$$

The linear, time-invariant system is stable if all roots of the transfer function denominator polynomial have negative real parts (i.e. are in the left half plane), and it is otherwise marginally stable or unstable. This is evidenced by examining the inverse Laplace transform of the general transfer function having no zeros:

$$L^{-1} \left\{ \frac{K}{(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_n)} \right\} = \sum_{i=1}^m k_i e^{p_i t}$$

The simplest form of the above is the first-order system  $G(s) = \frac{1}{s+a}$ , having impulse and step response for time  $t > 0$ :

$$L^{-1} \left\{ \frac{1}{s+a} \right\} = e^{-at} \quad L^{-1} \left\{ \frac{1}{s} \frac{1}{s+a} \right\} = (1 - e^{-at})$$



**Figure 5.1 – First-order system impulse response plot**

These responses converge exponentially for negative real poles, and grow exponentially for positive poles. Poles that are farther to the left in the left half plane have a faster speed of response.

Next consider the second-order system having complex roots:

$$G(s) = K \left\{ \frac{1}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)} \right\}$$

Manipulating the denominator polynomial produces:

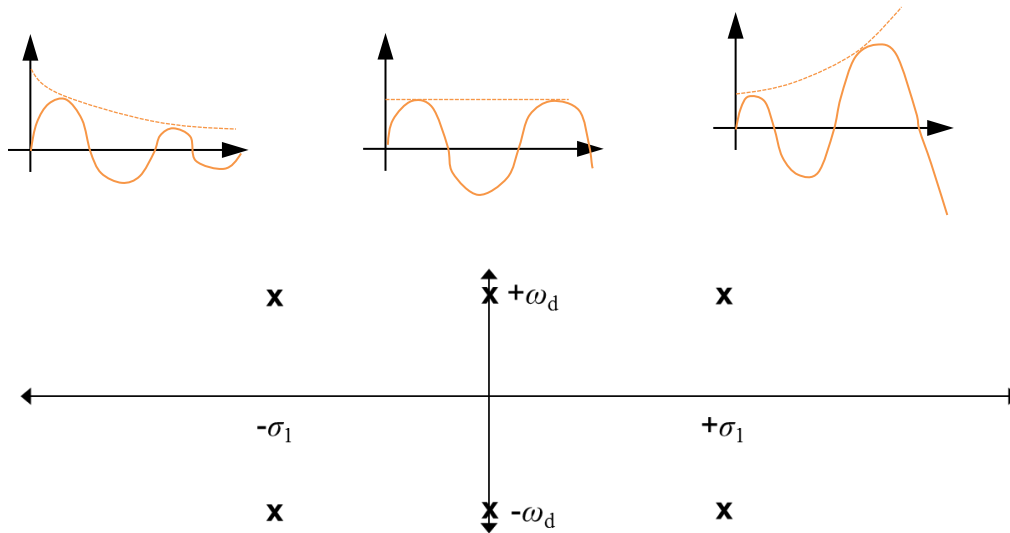
$$G(s) = \frac{\omega_n^2}{s^2 + 2\sigma s + (\sigma^2 + \omega_d^2)} \tag{5.1}$$

which after additional simplification and inverse Laplace transforming produces the impulse response:

$$L^{-1} \left\{ \frac{1}{(s + \sigma)^2 + \omega_d^2} \right\} = \frac{1}{\omega_d} e^{-\sigma t} \sin(\omega_d t)$$

valid for  $t > 0$ . Time domain responses are shown in Figure 5.2:



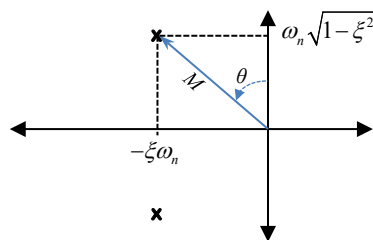


**Figure 5.2 – Second-order system time response plot**

Expressing the transfer function in an equivalent form involving the damping factor  $\zeta$  (zeta) and undamped natural frequency  $\omega_n$  :

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (5.2)$$

Comparison of Equations 5.1 and 5.2 shows that:  $\sigma = \zeta\omega_n$  and  $\omega_n^2 = (\sigma^2 + \omega_d^2) = (\zeta^2\omega_n^2 + \omega_d^2)$ , or that  $\omega_d = \omega_n\sqrt{1-\zeta^2}$ . This is depicted in Figure 5.3:



**Figure 5.3 – Poles of a 2<sup>nd</sup> order under-damped system**

Computing the magnitude  $M$  and angle  $\theta$ :

$$M = \sqrt{(\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \omega_n$$

$$\theta = \sin^{-1}(\zeta)$$

Thus as one increases  $\omega_n$  given a fixed  $\zeta$ , the poles move along straight lines, and for a given  $\omega_n$  as one increases  $\zeta$ , the poles move along the arc of a circle. The system is described to be undamped if  $\zeta=0$ , underdamped if  $\zeta<1$ , critically damped when  $\zeta = 1$ , which occurs when  $\theta = 90$  degrees, and overdamped when  $\zeta > 1$ .

The step response of this 2<sup>nd</sup> order system given by the inverse Laplace Transform:

$$L^{-1} \left\{ \frac{1}{s} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = 1 - \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos(\omega_d t - \phi)$$

$$\phi = \tan^{-1} \left( \frac{\xi}{\sqrt{1-\xi^2}} \right) \quad (5.3)$$

This time domain response is used to compute the rise time, settling time, and percentage overshoot PO for a general 2<sup>nd</sup> order system. Rise time is defined as the time for the unit step response to transition from 0.1 to 0.9. It can be determined by solving the closed-form step response for the times that it reaches 0.1 and 0.9 and computing the difference. This has been determined numerically and recorded in Table 5.1 below as a function of damping factor  $\xi$ .

Table 5.1 – Normalized rise time ( $T_r \omega_n$ ) versus damping factor  $\xi$

$\xi$	$T_r \omega_n$
0.1	1.10
0.2	1.20
0.3	1.32
0.4	1.46
0.5	1.64
0.6	1.85
0.7	2.13
0.8	2.47
0.9	2.88

The Control Handbook<sup>1</sup> gives a quadratic approximation to the normalized rise time  $T_r \omega_n = 2.230\xi^2 - 0.078\xi + 1.12$ , and a fairly good linear approximation is given by  $T_r \omega_n = 2.5\xi + 0.6$ .

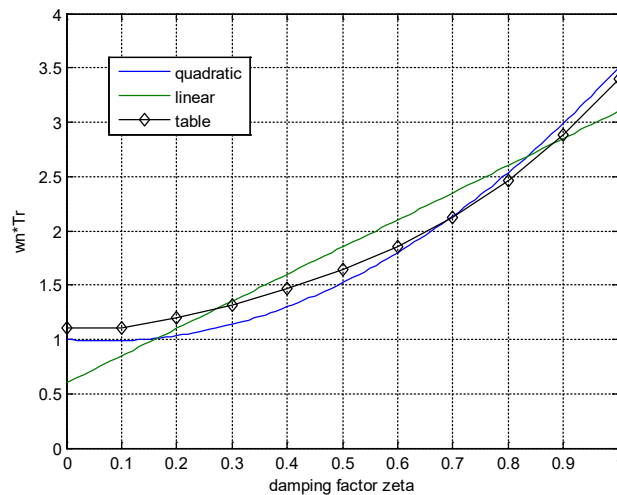


Figure 5.4 – Normalized rise time versus damping

<sup>11</sup> Levine, William S. [ed.], *The Control Handbook*, CRC Press Inc., 1996



Figure 5.4 shows that both the quadratic and linear approximations are good approximations of the actual results given in the table. However, an adequate approximation that is recommended is to use

$$\omega_n T_r = 1.8$$

an approximation that is sufficiently close to these curves in the range  $0.2 < \zeta < 0.8$ . Given this, then one can immediately compute the required undamped natural frequency required to achieve a specified rise time. For example, to achieve a rise time of 1 second or less, the natural frequency must be 1.8 rad/sec or more. Similarly, if a rise time of 1  $\mu$ sec is desired, a natural frequency of  $1.8 \times 10^6$  rad/sec is the goal. The dominant complex poles providing that response time are required to be outside and to the left of the circle on the complex plane having a radius of  $\omega_n$ .

Settling time is defined as the time for the response to reach and remain within 2% of its final value. It is determined by setting Equation 5.3 to 0.98 and solving for time. This yields  $T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$ ; again this is not that amenable to quick interpretation or easy usage. However for the range  $0.1 < \zeta < 0.7$  the numerator  $-\ln(0.02\sqrt{1-\zeta^2})$  is fairly close to the value 4 (see Figure 5.5).

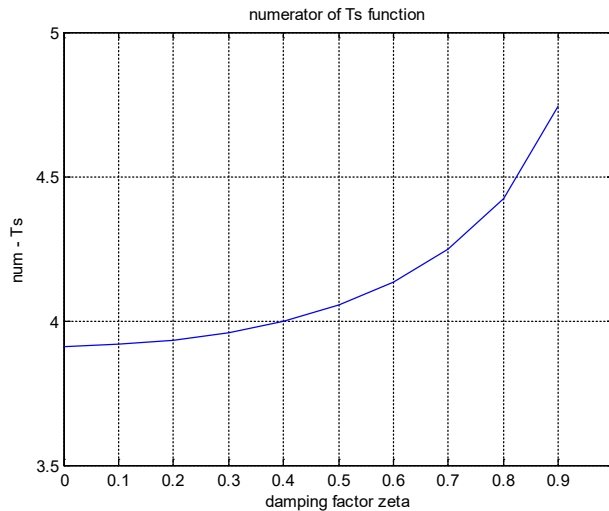


Figure 5.5 --  $-\ln(0.02\sqrt{1-\zeta^2})$  vs  $\zeta$

so an acceptable approximation is:

$$T_s = \frac{4}{\zeta\omega_n}$$

The percentage overshoot is defined as the degree that the response overshoots the final value. This is computed by solving for the time that the response reaches its peak (where the derivative, or slope, is zero), and then computing the response value at that time. This is divided by the final value and scaled by 100 to yield a percentage. The result:  $PO = 100 \exp\left(-\pi\zeta / \sqrt{1-\zeta^2}\right)$ . A system having a % overshoot that is less than a specified amount will have a damping factor  $\zeta$  that is greater than that given by this equation.



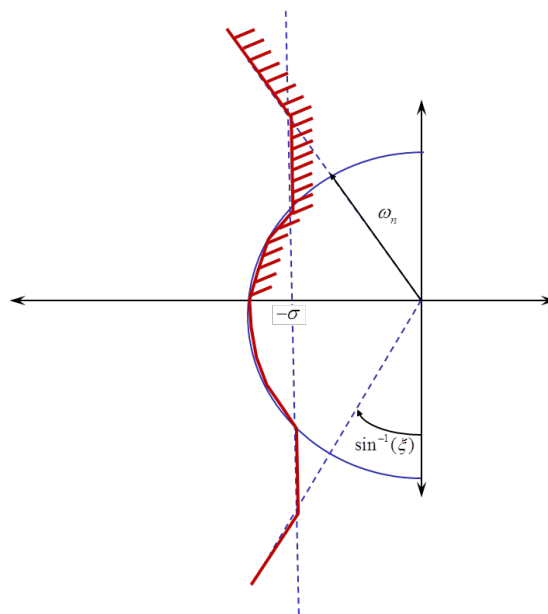
Summarizing these three time response relationships:

$$T_r \approx \frac{1.8}{\omega_n}$$

$$PO = 100 \exp\left(-\pi\xi / \sqrt{1-\xi^2}\right)$$

$$T_s \approx \frac{4.0}{\xi\omega_n} = \frac{4.0}{\sigma}$$

Thus, to satisfy specifications constraining the rise time, settling time, and percentage overshoot in a second order system, one needs to produce a system having poles remain outside the circle of radius  $\omega_n$ , to the left of the vertical line that crosses the real axis at  $-\sigma$ , and to the left of the lines emanating from the origin at an angle of  $\sin^{-1}(\xi)$ . This is depicted in Figure 5.7



**Figure 5.6 – Dominant pole location constrains meeting rise time, settling time, and PO relationships**

**Example** – Determine the damping factor  $b$  and spring rate coefficient  $k$  for a 2<sup>nd</sup> order spring-mass-damper system, such that the eigenvalues of the system (i.e. poles) provide the system with a rise time of 0.5 seconds, a percentage overshoot of 20% or less, and a settling time of 2 seconds. The mass of the system is 1 kg. Use the approximate equations for rise time and settling time prediction ( $1.8 / \omega_n$  and  $4 / \sigma$ ). The transfer function is:

$$\frac{Y(s)}{U(s)} = \frac{k}{ms^2 + bs + k}$$

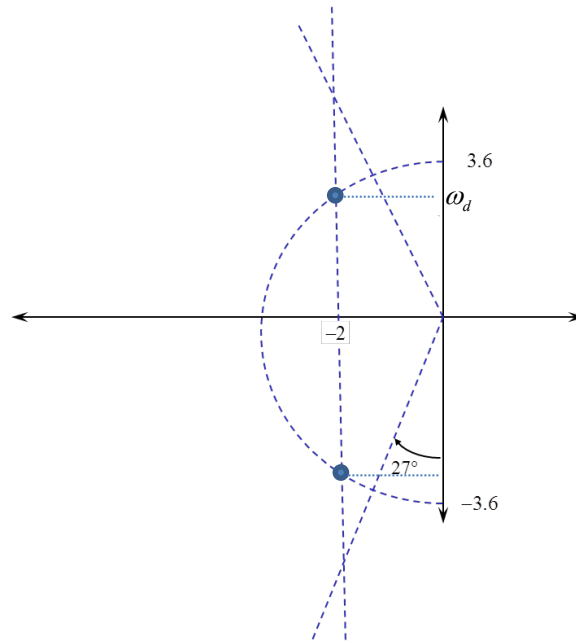
Solution – The characteristic equation is given by:  $s^2 + (b/m)s + (k/m) = s^2 + 2\xi\omega_n s + \omega_n^2$ . The rise time places a lower constraint on  $\omega_n$ :

$$\omega_n > \frac{1.8}{0.5} = 3.6 \text{ rad/sec}$$

The settling places a lower constraint on  $\sigma$  :

$$\sigma > \frac{4}{2} = 2 \text{ rad/sec}$$

The percent overshoot constraint requires that  $\xi$  be greater than that given by solving the PO equation with PO = 20:  $\ln(0.2) = -\pi\xi / \sqrt{1-\xi^2}$  or  $\xi > 0.46$ . The angle corresponding to this is  $\sin^{-1}(\xi) = 27$  degrees, which is shown in Figure 5.7.



**Figure 5.7 – Solution to Example Problem**

We can place the poles anywhere to the left of these lines. As the figure shows, the poles are placed at  $-2 \pm j\omega_d$ . Thus they will result in a response that meets the rise time and settling time requirements, and does better than the overshoot requirement, providing less than a 20% overshoot. At this location the damping ratio is higher –  $\xi = \sin \theta = 2 / 3.6 = 0.56$  – and therefore the PO will be less than the 20% limit.

Solving for  $\omega_d = \omega_n \sqrt{1-\xi^2} = 3.6 * \text{sqrt}(1-.56^2) = 3.0 \text{ rad/sec}$ . The poles are at  $-2 \pm j3$ .

Equating the coefficients of the transfer functions:

$$k = (\omega_n)^2 m \quad \text{and} \quad b = 2\xi\omega_n m$$

Plugging in these values identified for  $\omega_n$  and  $\xi$  :  $k = (3.6)^2 1 = 12.96 \text{ N/m}$  and  $b = 2\xi\omega_n m = 2\sigma m = 2 \cdot 2 \cdot 1 = 4 \text{ N/(m/s)}$ .

## HAND PLOTTING OF BODE PLOTS

In order to be able to extract an appropriately matching transfer function from frequency response data, it is useful to first learn the rules for plotting frequency response curves from transfer functions.

### CURVE PLOTTING RULES

#### 1. Manipulate the transfer function (TF) into Bode form

- a. and for each factored term, both the 1<sup>st</sup> and 2<sup>nd</sup>, divide through by the DC coefficients multiplying  $s^0$

e.g. 
$$K \frac{s/a + 1}{(s^2/\Omega^2 + 2\xi s/\Omega + 1)(s/b + 1)}$$

#### 2. Plot Magnitude Curve:

- Determine value of  $n$  associated with the poles or zeros at the origin (class 1 terms)
- Sketch the straight line approximation to which the magnitude converges asymptotically for the low frequency region:
  - This line passes through the value  $K$  at frequency  $\omega=1$  rad/sec.
- Working from low to high frequency, identify the breakpoints, then step the slope by  $\pm 1$  or  $\pm 2$ , depending on whether it is a 1<sup>st</sup> or 2<sup>nd</sup> order term in the numerator (+) or denominator (-).
  - Start the next line segment at the point on the straight line approximation to the left at the break point.
- Sketch in the actual magnitude curve, running along the straight line approximations except at the break points:
  - at 1<sup>st</sup> order breaks increase value by 1.4 (+3dB) for numerator breaks, and decrease by 0.707x (-3dB) at denominator breaks
  - at 2<sup>nd</sup> order breaks, sketch in resonant peaks
    - Use  $G(j\omega)=1/2\xi$  for denominator breaks,  $G(j\omega)=2\xi$  for numerator breaks

#### 3. Plot Phase Curve:

- Plot low frequency asymptote phase curve  $\Phi = n \times 90^\circ$
- Plot high frequency asymptote phase curve  $\Phi = -(m-k) \times 90^\circ$ 
  - where  $m$  = order of denominator
  - when  $k$  = order of the numerator
- Sketch in phase curve approximations by changing phase by  $\pm 90^\circ$  for 1<sup>st</sup> order breaks, or  $\pm 180^\circ$  for 2<sup>nd</sup> order breaks
  - For 1<sup>st</sup> order breaks, sketch a straight line approximation running from  $0.2\omega_B$  to  $5\omega_B$
  - Sketch in the actual phase by adding +11 deg and -11 deg at the  $0.2\omega_B$  to  $4\omega_B$  locations
  - For 2<sup>nd</sup> order breaks, sketch in actual phase curves from approximates using 2nd order phase plot

**Example 1** – Plot by hand the Bode plots of the following transfer function:

$$\frac{20000s + 10000}{s^3 + 60s^2 + 500s}$$





1 - Convert to Bode form:

$$\frac{20000(s + 0.5)}{s(s + 10)(s + 50)}$$

Divide through by DC coefficients

$$20000 \frac{0.5(s/0.5 + 1)}{10 * 50 * s(s/10 + 1)(s/50 + 1)} = 20 \frac{(s/0.5 + 1)}{s(s/10 + 1)(s/50 + 1)}$$

2 - Class 1 term:  $\frac{20}{s}$

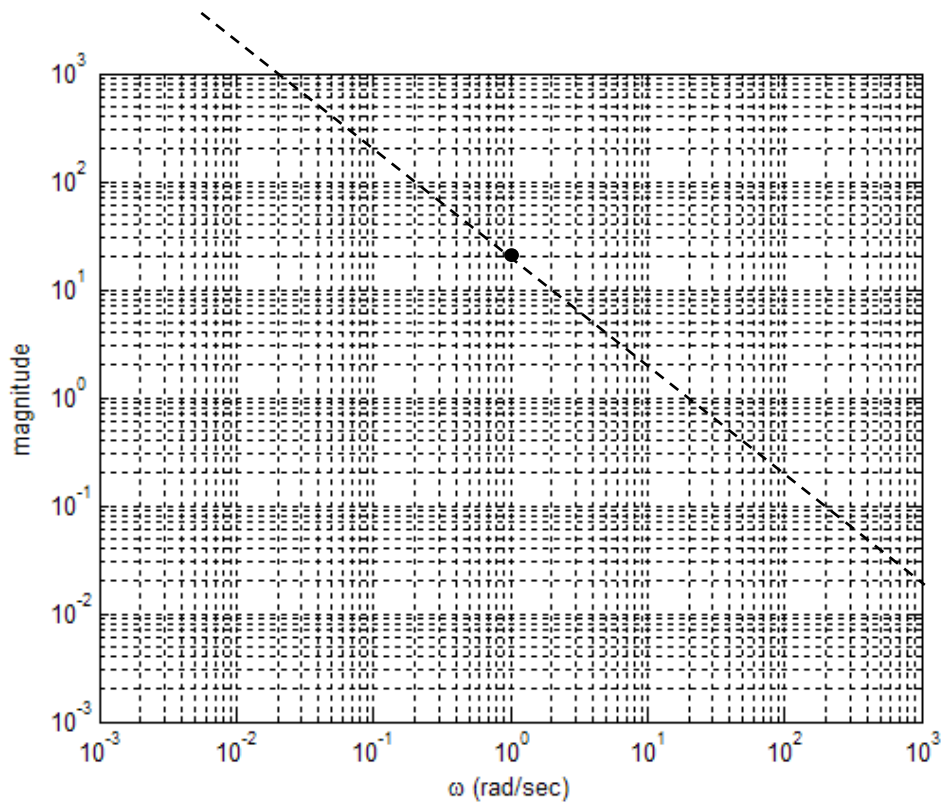
2a -

$$n = -1;$$

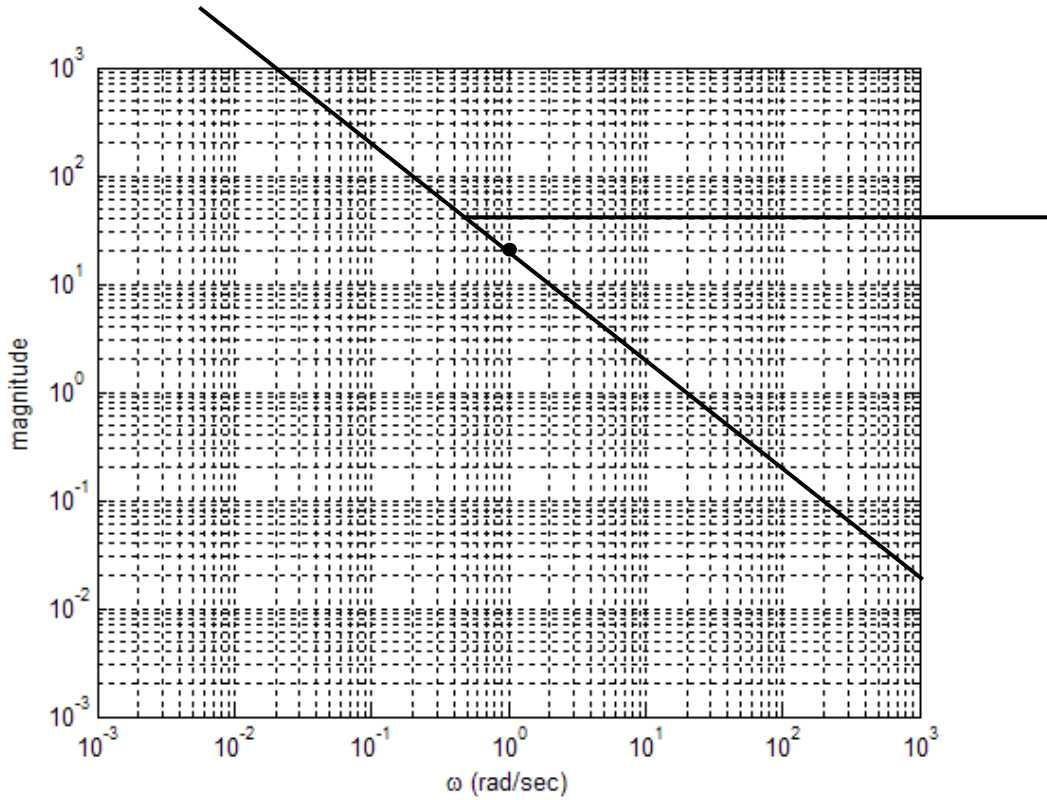
$$K = 20$$

2b - Sketch in the straight line approximation for 20/s.

Note that it goes through the point  $G(j1) = K$ :



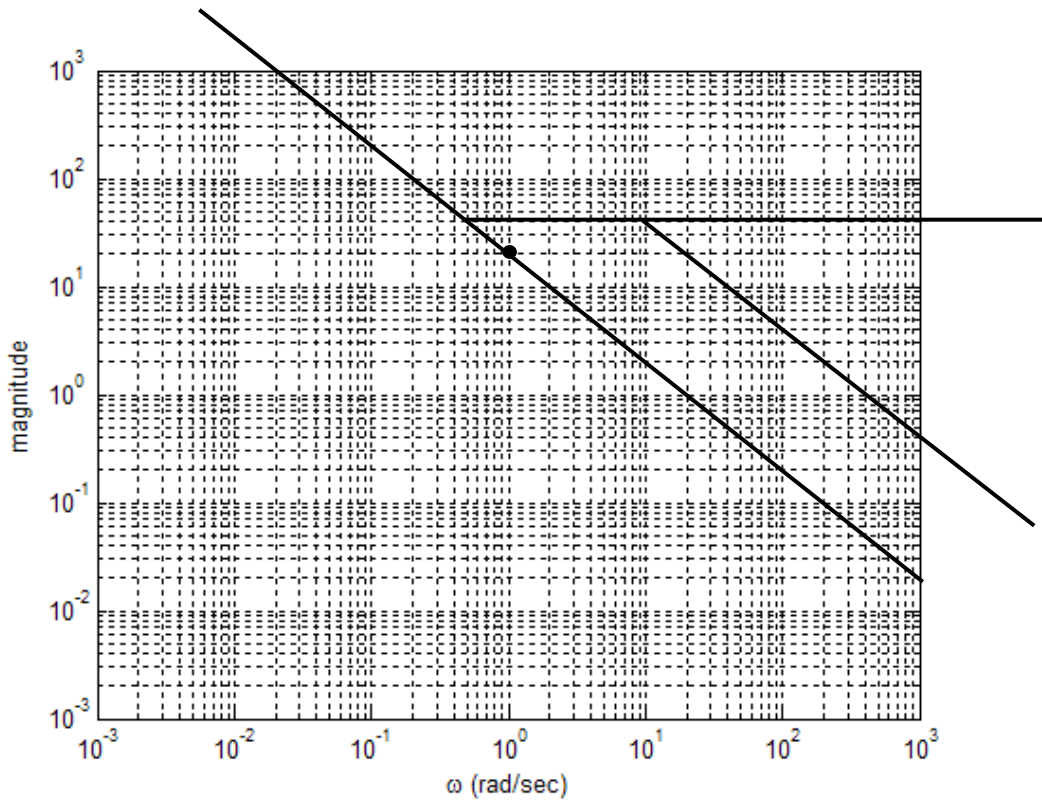
2c – Add the asymptotic approximate line for the first break. Note that there are 3 break frequencies; 20, 50, and 0.5 rad/sec. Starting at the lowest – 0.5 rad/sec.



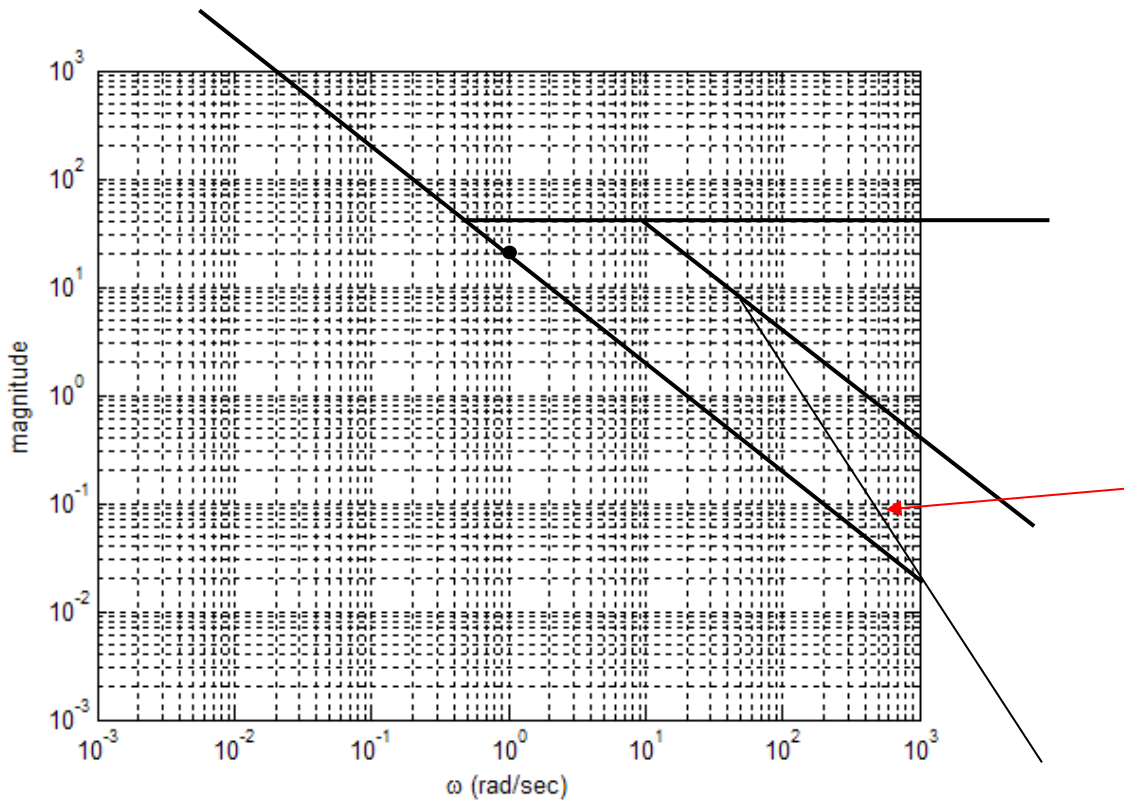
Note that the slope changes from -1 to 0 (i.e. from -20 dB/decade to 0).

Next add the line for the pole at 10 rad/sec, adding an addition slope of -1:



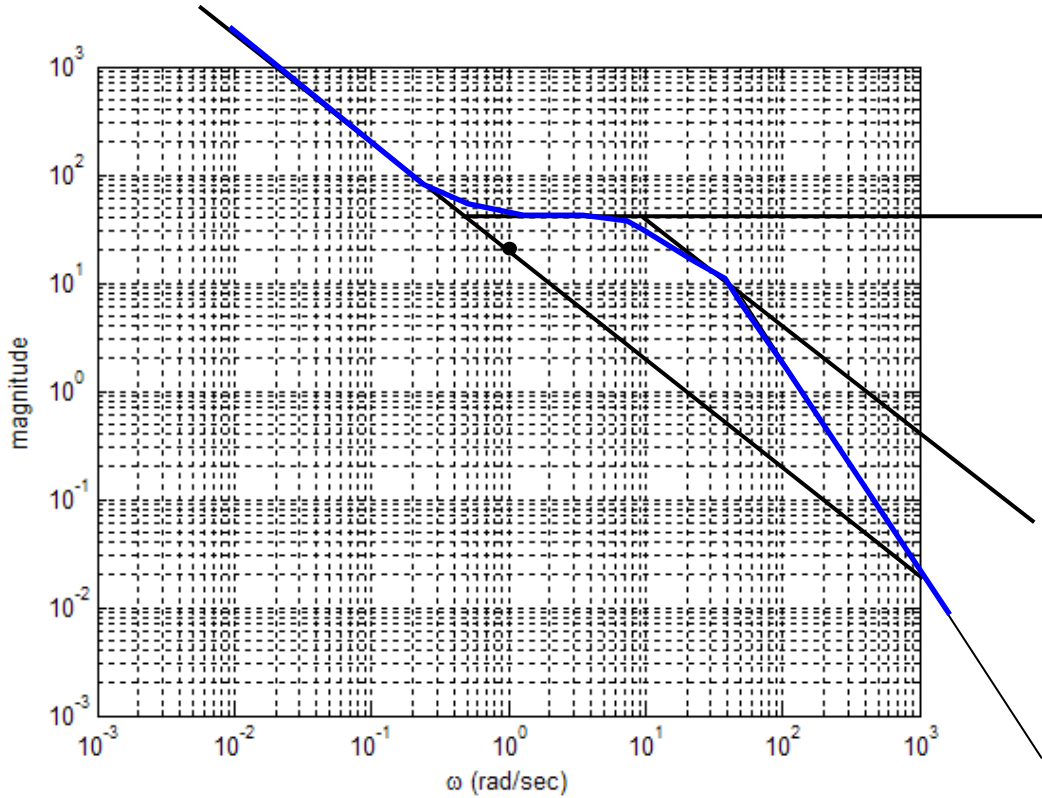


Finally the last pole at 50, changing the slope from -1 to -2 (or -40 dB/decade):



This is done by noting that on the straight line approximation starting at the break of 10 rad/sec, the gain value at 50 rad/sec is about 8 (absolute). Thus when the new line what starts at the break of 50 rad/sec gets to 500 rad/sec it will have a value of 8/100, 2 decades down, or 0.08. This is indicated by the red arrow.

**2d** – sketching in the actual magnitude curve:



3

**Plot Phase Curve:**

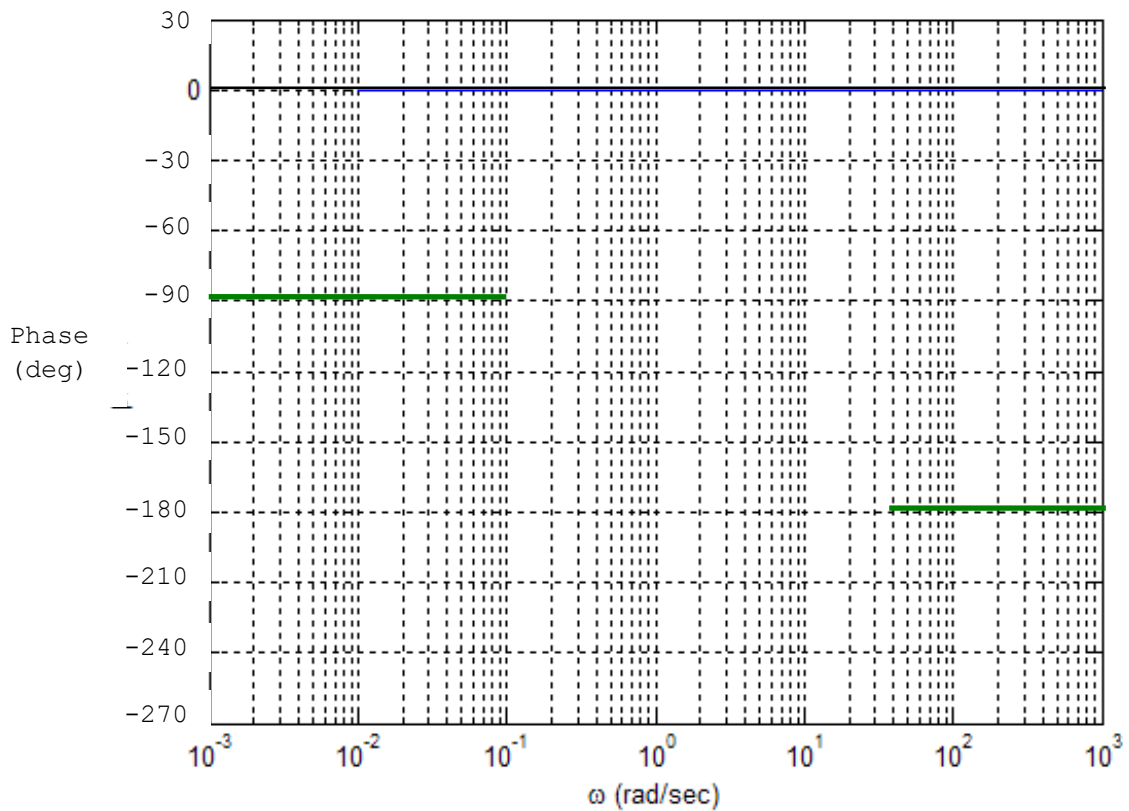
(a) Plot low frequency asymptote phase curve  $\Phi = n \times 90^\circ$

- where  $n = -1$

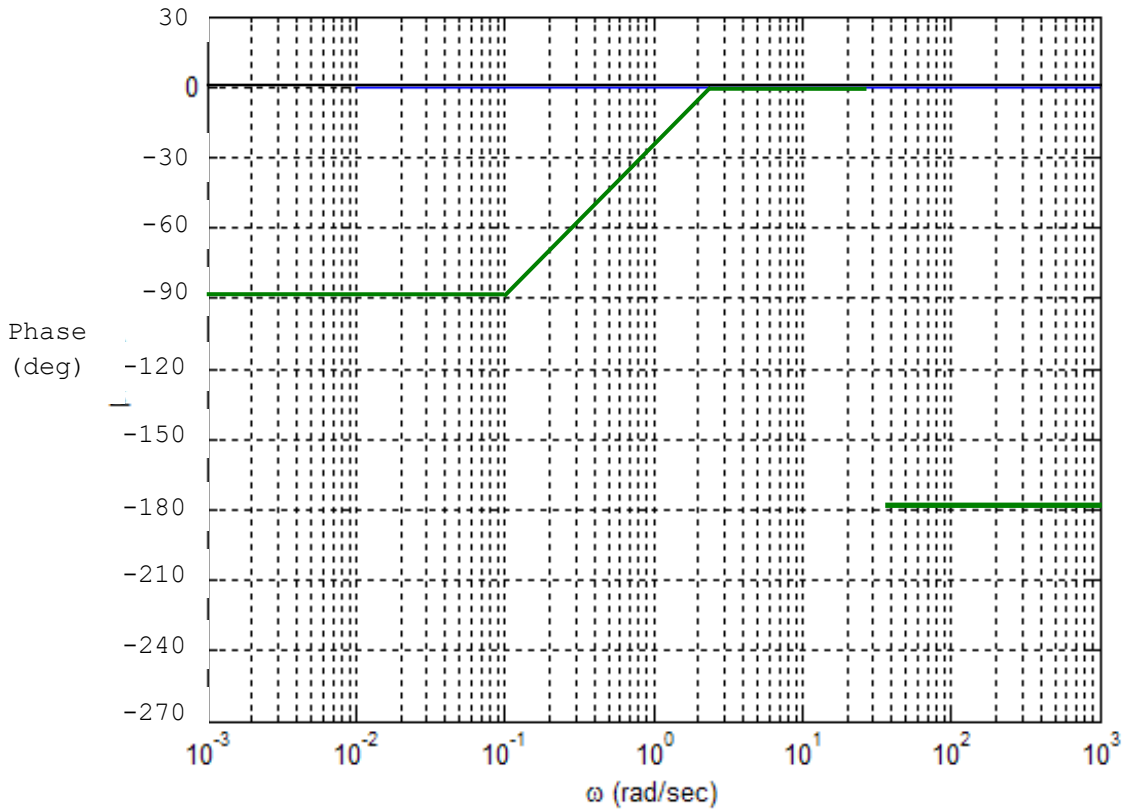
(b) Plot high frequency asymptote phase curve  $\Phi = -(m-k) \times 90^\circ$

- where  $m = \text{order of denominator} = 3$
- when  $k = \text{order of the numerator} = 1$

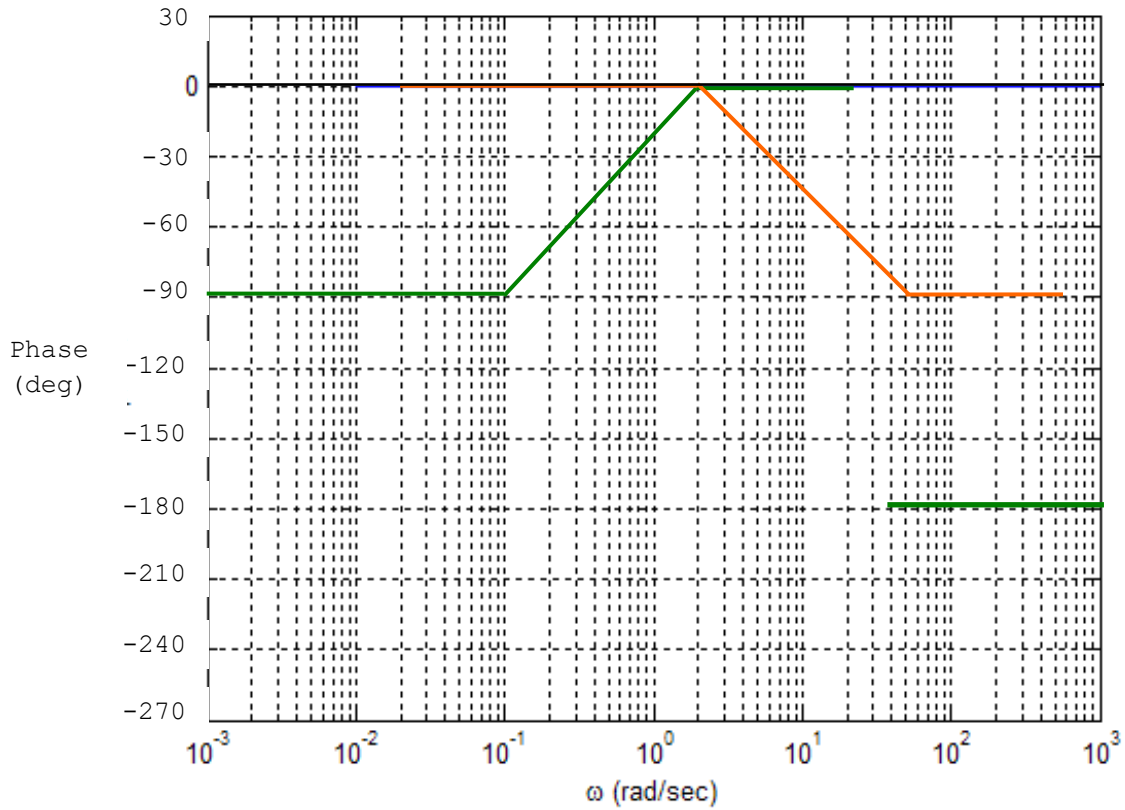
In this example  $n = -1$  so the phase curve will start at the left at  $-90^\circ$  and end on the right at  $-180^\circ$ .



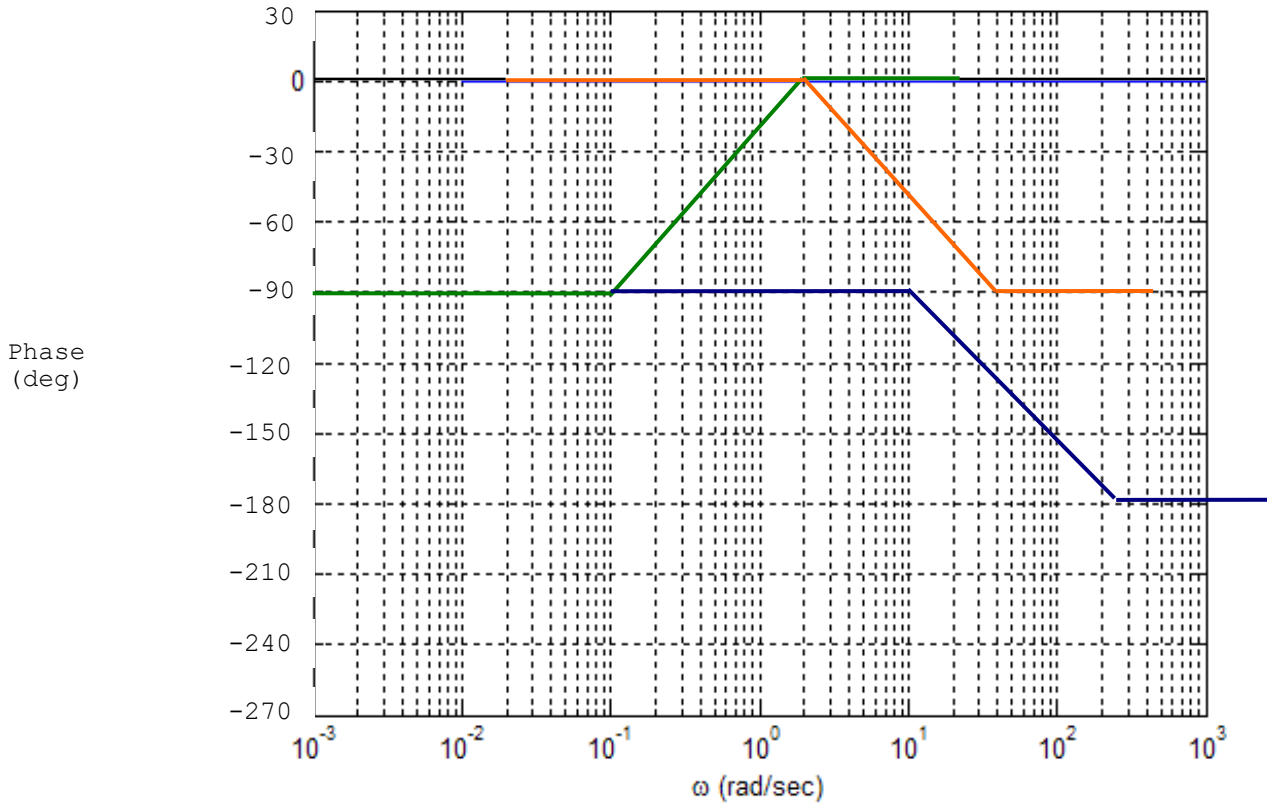
Next sketch the phase change due to the first order zero. Note that the straight line approximation starts at  $0.2 \cdot 0.5$  and ends at  $5 \cdot 0.5 = 2.5$  as shown here.



**3c** -- Next account from the break at 10 which is a pole contributing an additional -90 degrees. This straight line approximation is shown in orange. We begin the approximation from 0 degrees since that is the end point for the previous break:

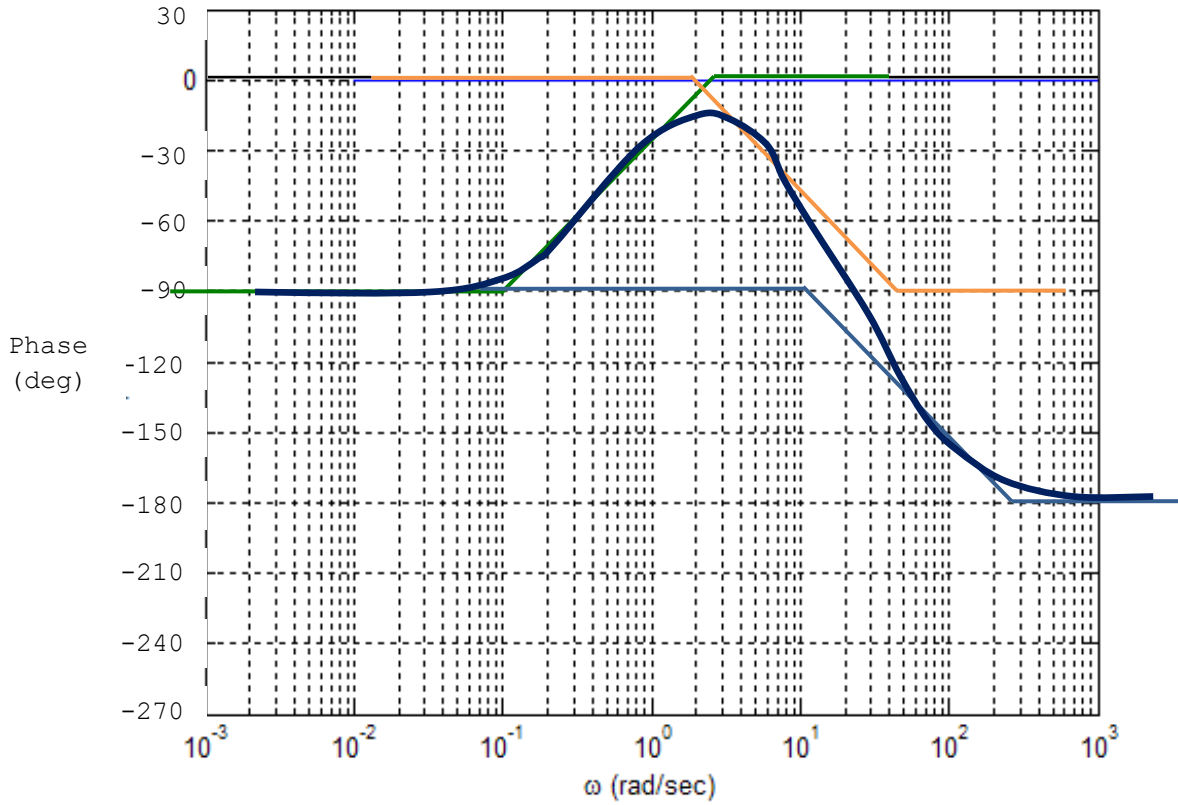


Finally adding the last additional phase shift of -90 for the break to 50 rad/sec. We begin the approximation from -90 degrees since that is the end point for the previous break:

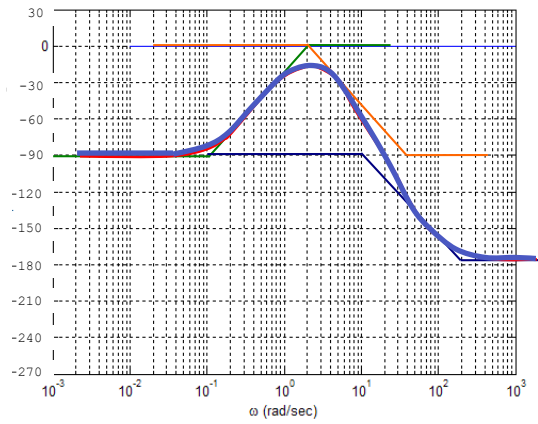
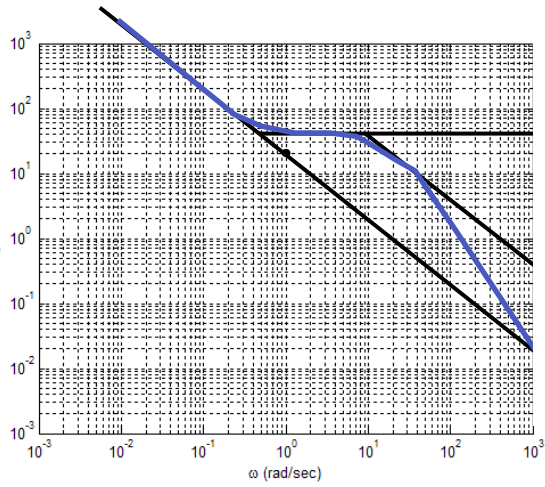
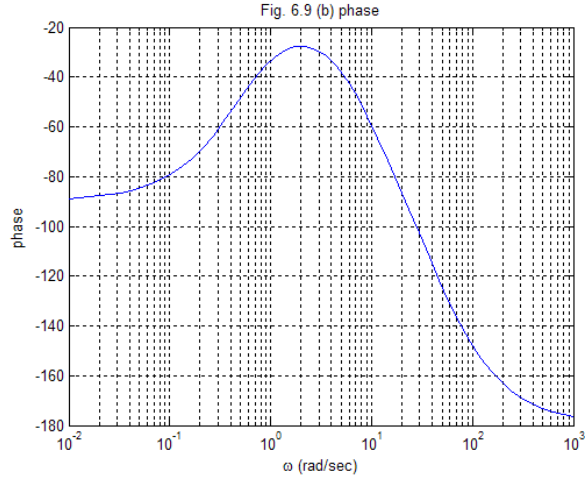
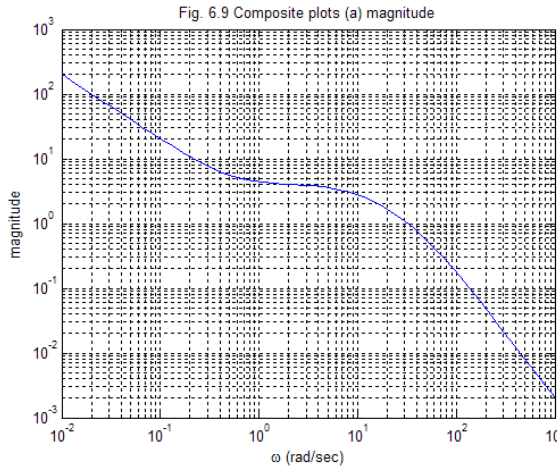




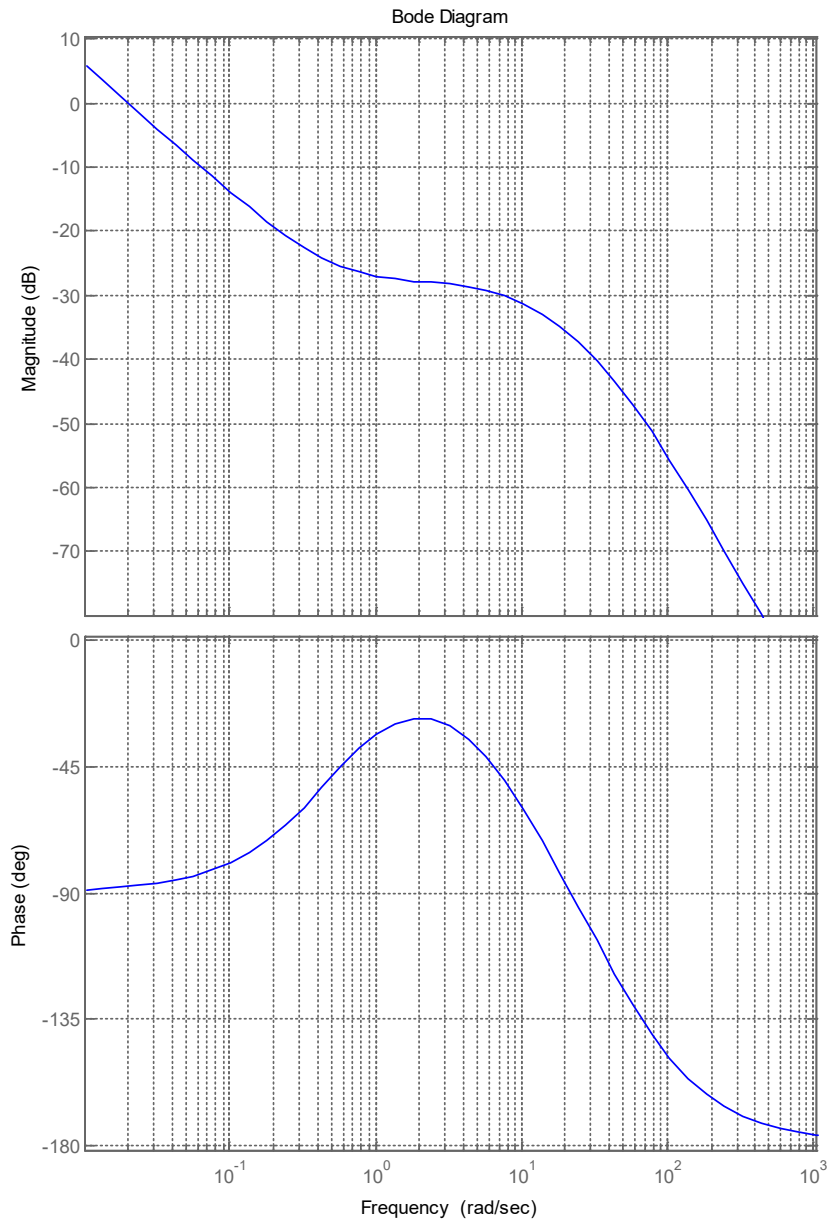
Sketching in the composite phase curve:



Finally, comparing the approximate hand drawn to the curves plotted with Matlab:



Replotting with units of dB instead of absolute magnitude:

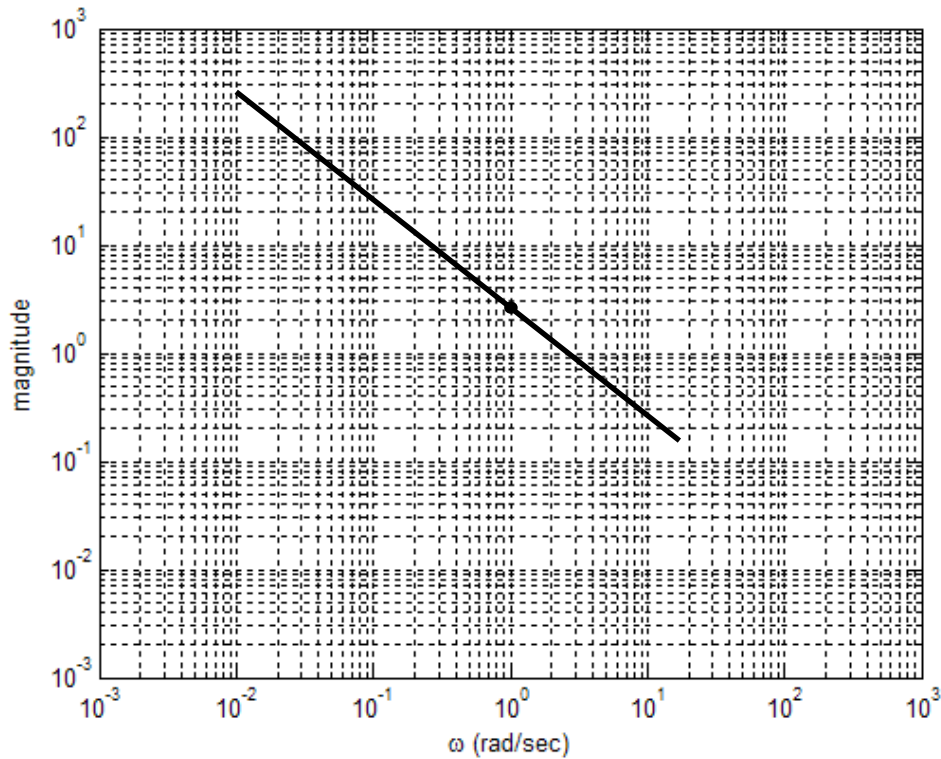


Example 2 --

$$G(s) = \frac{10}{s(s^2 + 0.4s + 4)}$$

$$G(s) = \frac{10}{4} \frac{1}{s(s^2/4 + 0.1s + 1)}$$

Class 1 slope and gain:  $n = -1$ ;  $K = 2.5$



Complex pole pair:

$$\frac{1}{(s^2/4 + 0.1s + 1)} = \frac{1}{(s^2/\Omega^2 + 2\xi s/\Omega + 1)}$$

thus  $\Omega = 2$  rad/sec and  $\xi = 0.1$ .

Natural frequency = 2 rad/sec. At that frequency ( $\omega = \Omega$ ) the resonant peak is given by:

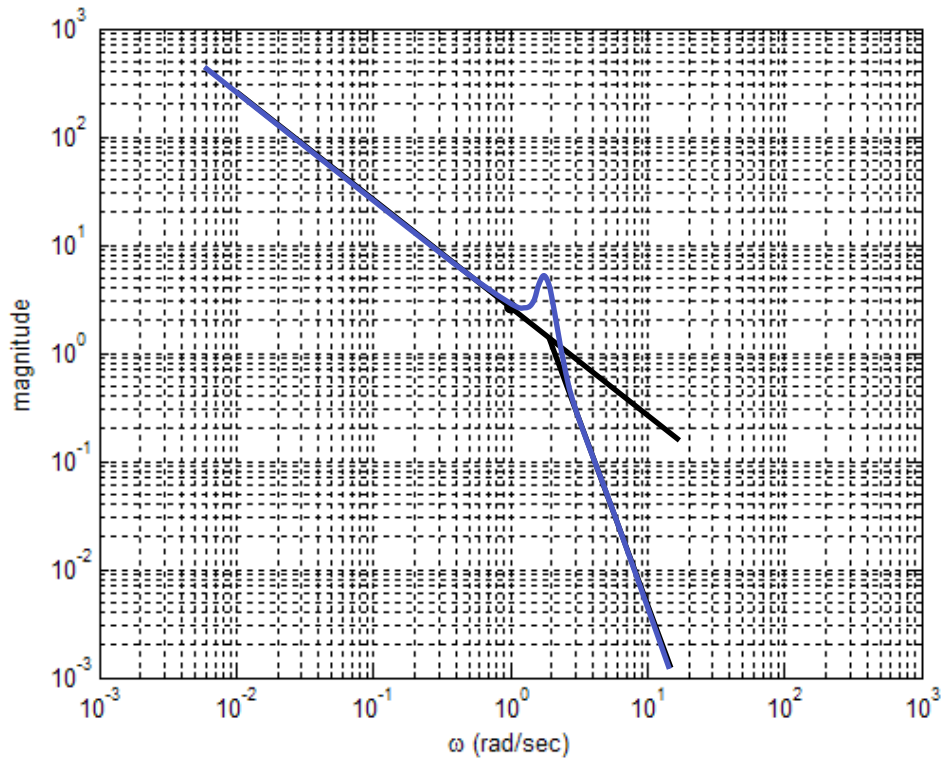
$$\left. \frac{1}{(s^2/\Omega^2 + 2\xi s/\Omega + 1)} \right|_{s=j\omega} = \frac{1}{(-1 - 2\xi j\omega/\Omega + 1)} = \frac{1}{2 * 0.1 * j\omega/\Omega} = \frac{1}{0.2 * j}$$



Taking the magnitude:

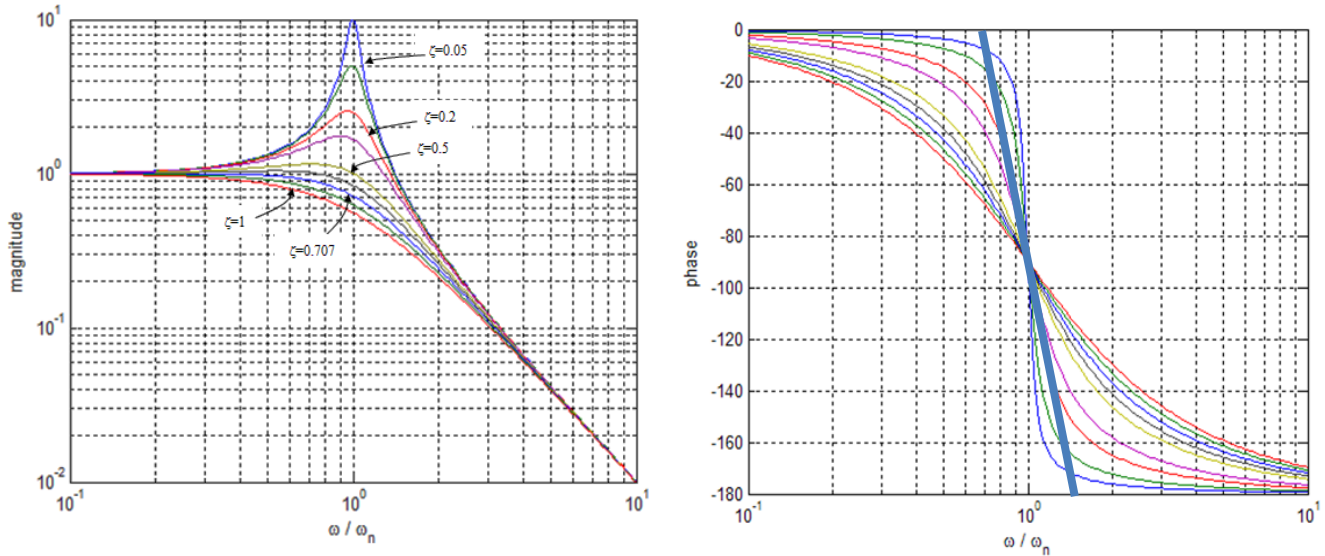
$$\left| \frac{1}{0.2*j} \right| = 5$$

Thus the resonant peak will provide a gain of 5x at  $\omega = 2$ . This is followed by a slope change of -2 from that frequency and to the right.

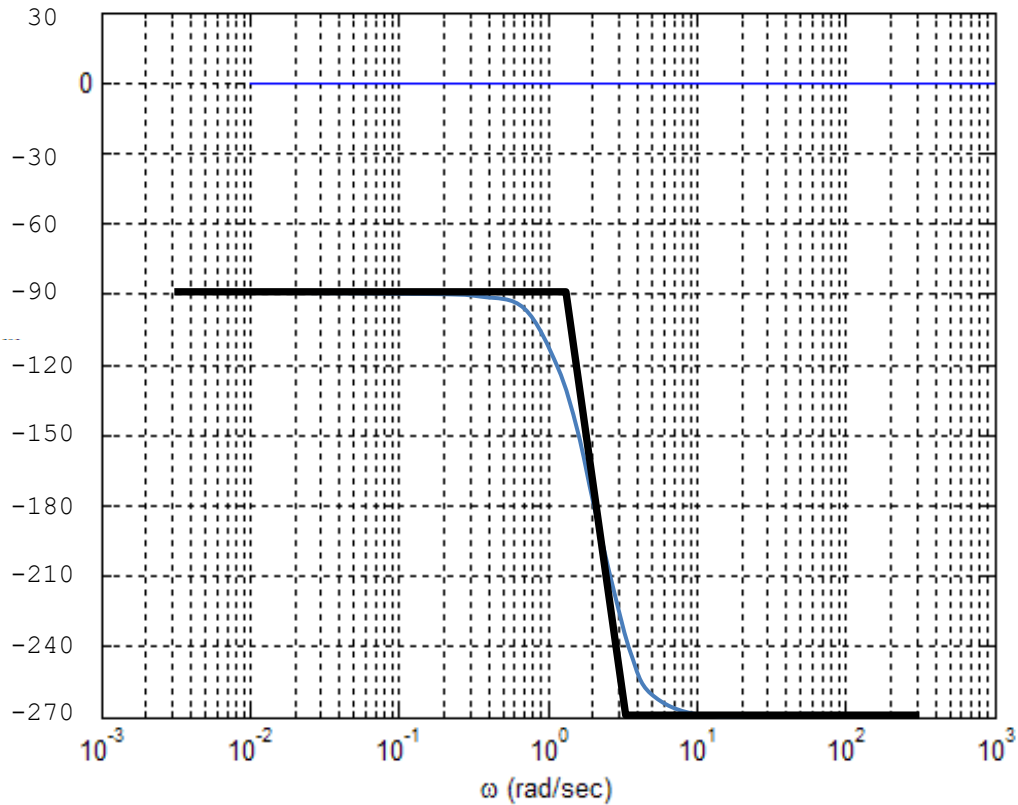


A gain of 5x on a gain value of 1.4 at the intersection of these two curves gives a resonant peak of  $\sim 7$ .

For 2nd order breaks, we sketch in the phase curve using approximates using 2nd order phase plot family of curves described in the lecture:



In the case of the 2<sup>nd</sup> order phase curve there is not one straight line asymptote that can be used to approximate the change in phase. I've drawn in a straight line that follows the red (zeta = 0.2) line at the break frequency. Use this to sketch in the approximate phase curve.

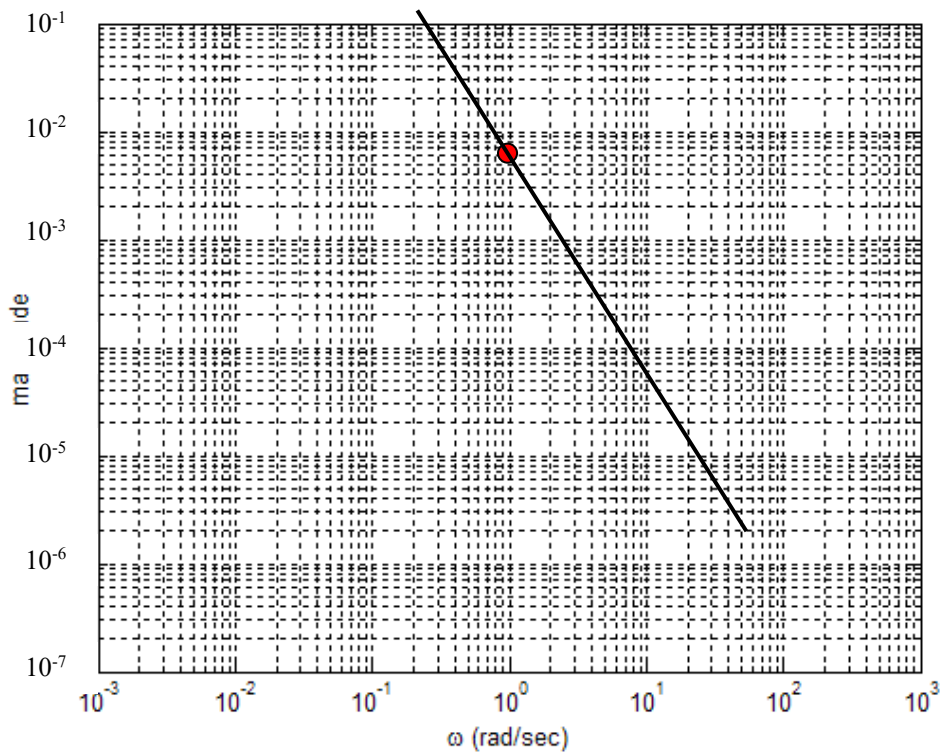


Example 3 --

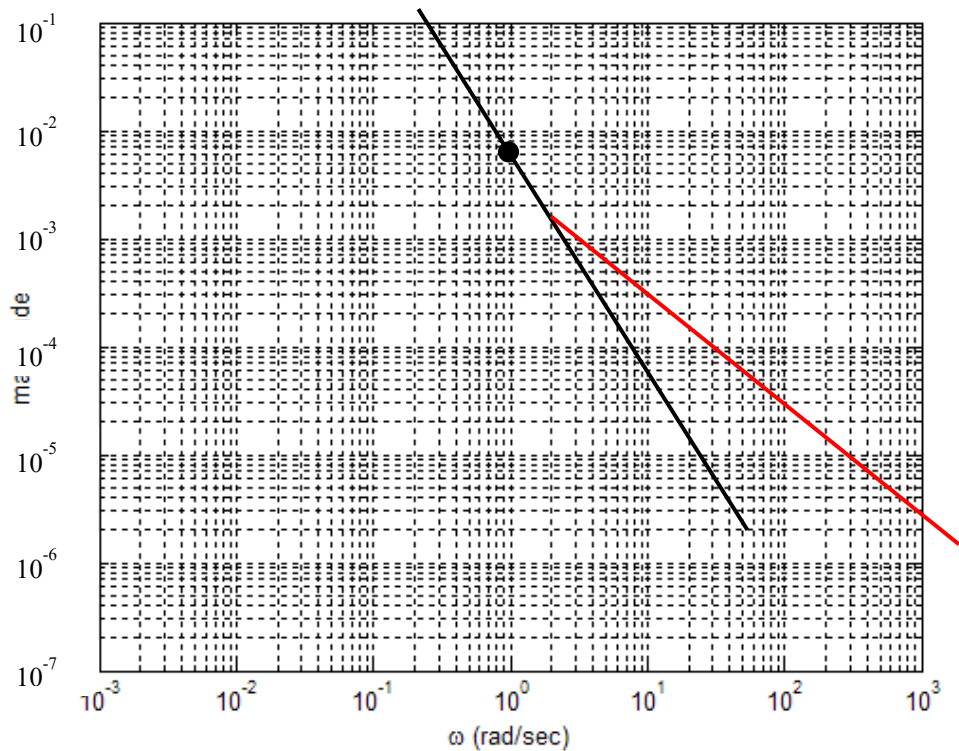
$$G(s) = \frac{(s + 2)}{s^2(s + 10)(s^2 + 6s + 25)}$$

$$G(s) = \frac{2}{10 * 25s^2} \frac{(s/2 + 1)}{(s/10 + 1)(s^2/5^2 + 6s/25 + 1)}$$

Class 1 slope and gain:  $n = -2$ ;  $K = 2/250 = 0.008$



Adding the impact of the zero at 2 rad/sec:



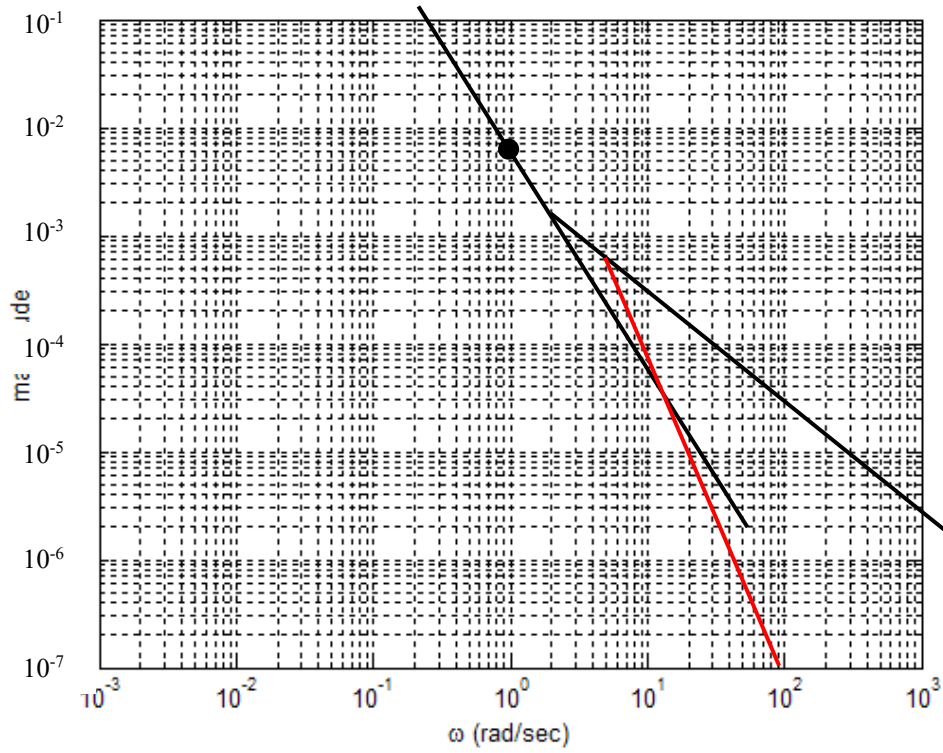
The next root as we move from left to right, the complex pole pair at 5 rad/sec:

$$\frac{1}{(s^2 / 5^2 + 6s / 25 + 1)} = \frac{1}{(s^2 / \Omega^2 + 2\xi s / \Omega + 1)}$$

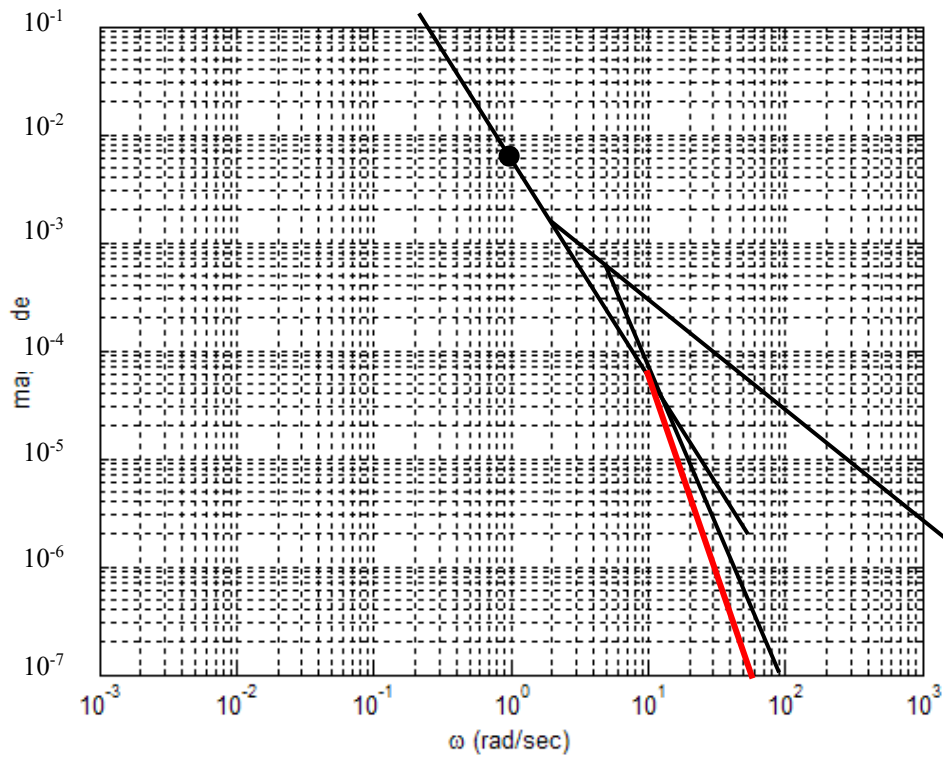
thus  $\frac{2\xi}{5} = \frac{6}{25}$ , and  $\xi = 0.6$ . In this case the resonant peak is less than 1, or approximately 1/1.2. Looking above at the 2<sup>nd</sup> order magnitude plot family, note that for  $\xi = 0.6$  there is no peak and that the gain at the break frequency to one decimal of precision is 0.8 (i.e. 1/1.2).



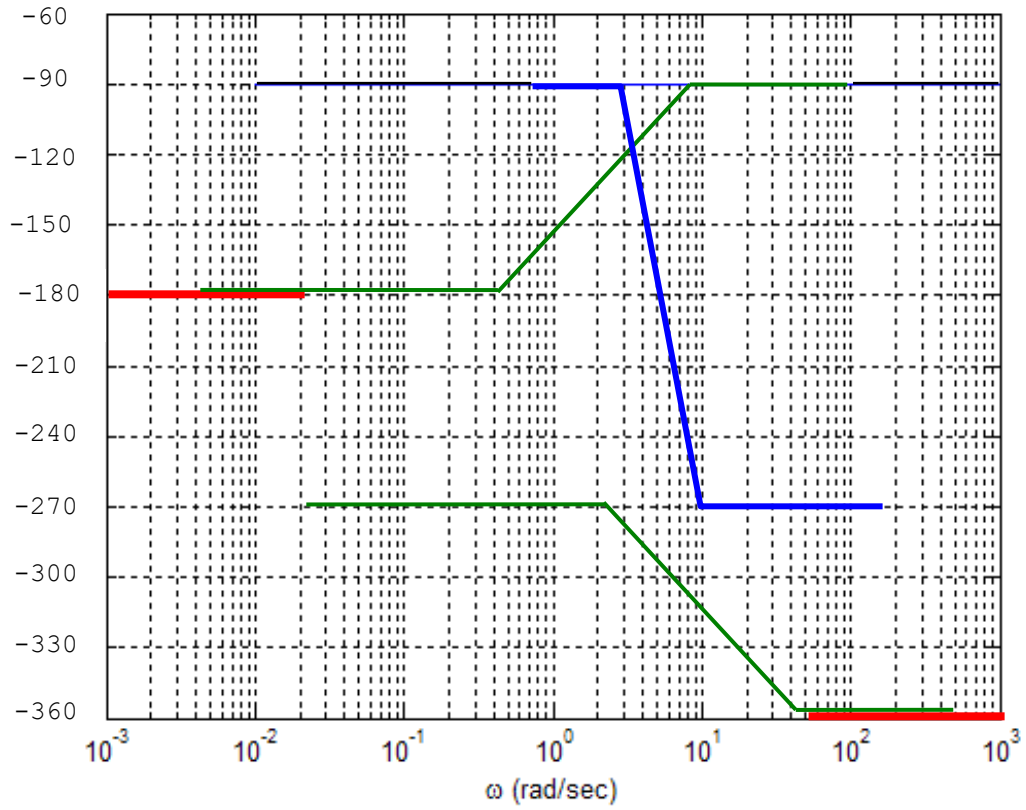




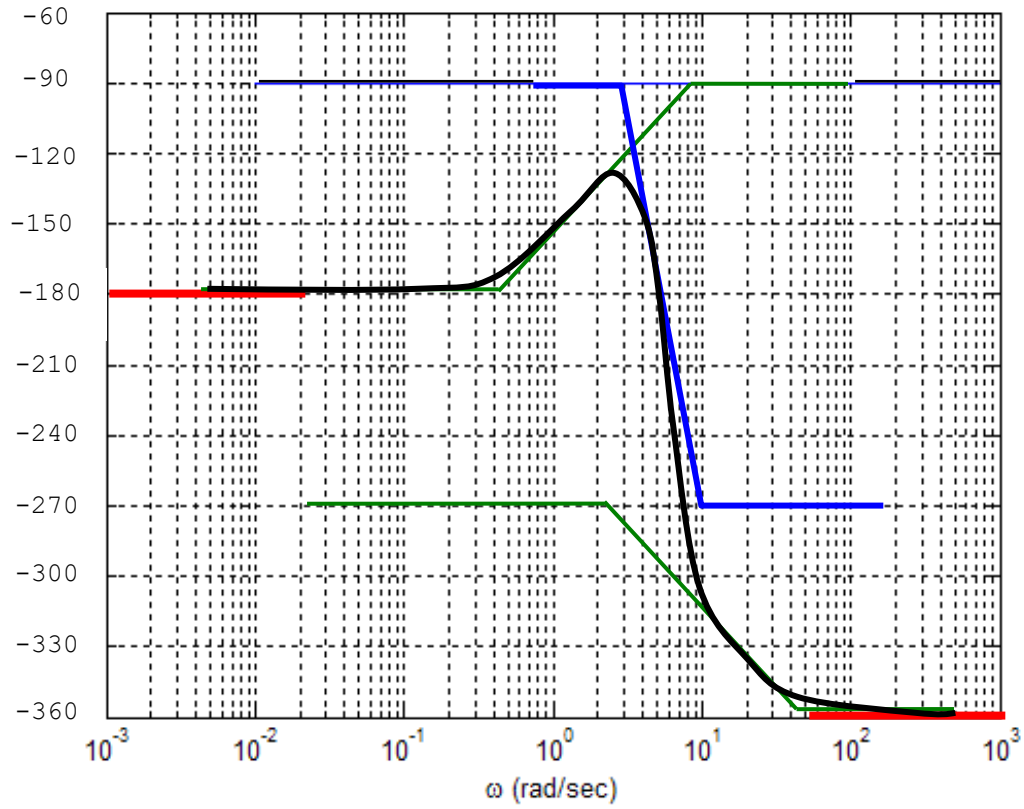
Finally the last pole at 10 rad/sec:



Sketching in the straight line approximations for phase associated with each pole and zero:



which enables us to sketch the composite phase response curve shown below in black:



## ADDITIONAL EXAMPLES

$$G(s) = \frac{10}{s(s^2 + 0.4s + 4)}$$

$$G(s) = \frac{1}{s(s+1)(s+5)(s+10)}$$

$$G(s) = \frac{(s+2)}{s(s+1)(s+5)(s+10)}$$

$$G(s) = \frac{(s+7)(s+2)}{s(s+1)(s+5)(s+10)}$$

$$G(s) = \frac{(s+2)}{s^2(s+10)(s^2+6s+25)}$$

$$G(s) = \frac{s}{(s+10)(s^2+6s+25)}$$

## STABILITY MARGINS

Stability in a single-input single-output system is assessed with the help of the following definitions:

- Crossover Frequency – the frequency where the loop gain magnitude curve crosses a gain of 1.
- Gain margin – the increase in loop gain that results in the system having a gain of 1 or greater at the frequency where the phase reaches -180 degrees.
- Phase margin – the negative of the phase change that would cause the phase at the crossover frequency to reach -180 degrees.